THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTION OF ELLIPTIC AND PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

by

© NANCY N. P. TING

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF ALBERTA

EDMONTON, ALBERTA

AUGUST, 1968

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTION OF ELLIPTIC AND PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS", submitted by NANCY N. P. TING, B.Sc., in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis deals with second order partial differential equations of elliptic and parabolic types containing a small parameter ε associated with the second order derivatives. Consider problems I and II below with operators L_ε and L_0, respectively. The differential operator L_ε is given by $L_\varepsilon = \varepsilon L_1 + L_0$ where L_0 is a differential operator of the first order and L_1 is a differential operator of the second order. If u_ε is the solution of problem I having PDE $L_\varepsilon[u_\varepsilon] = f$ satisfying the given boundary or initial conditions and u_0 is the solution of problem II having PDE $L_0[u_o] = f$ satisfying an appropriately modified initial condition, then, for ε small, we can argue that the two solutions should be quite similar, at least in some sense. The problem is to determine conditions under which $u_\varepsilon \to u_0$ as $\varepsilon \to 0$. In addition we examine how the difference between u_ε and u_0 behaves on those portions of the boundary where u_0 does not satisfy the prescribed condition on u_ε. Indeed the difference consists of two terms, the first of which is small (say $O(\varepsilon^n)$) uniformly in ε in the whole region of the problem and the second of which is appreciable only near the boundary but is (say) $O(e^{-\delta/\varepsilon})$ away from the boundary.

ACKNOWLEDGEMENTS

I would like to thank Dr. R. G. Sinclair for suggesting the topic and for his valuable guidance and advice. I would also like to thank the typists Mrs. G. Smith and Miss J. Talpash for their excellent job.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

Suppose L_0 is a first order linear partial differential operator whose coefficients are functions of x and y and L_1 is a second order linear partial differential operator with constant coefficients of either elliptic or parabolic type. Let

$$L_\varepsilon = \varepsilon L_1 + L_0 .$$

Suppose $u(x,y;\varepsilon)$ is the solution of the problem

$$L_\varepsilon[u] = f$$

in an open domain D subject to suitable boundary conditions. If we call $U(x,y)$ the solution of the associated reduced equation

$$L_0[U] = f$$

over the same domain D subject to the same boundary conditions as u on at least part of the boundary, then it has been shown that under certain conditions that

$$u(x,y;\varepsilon) \to U(x,y) \quad \text{as} \quad \varepsilon \to 0^+$$

uniformly over most of D. The regions in which this condition does not hold are to be found near the boundaries where the values of U and the arbitrarily defined values of u do not coincide. To accomodate this difference, a boundary layer term, called $z(x,y;\varepsilon)$

in this thesis, is introduced.

There is a rapidly growing literature dealing with this type of problem. The general survey of the work done on this subject before 1955 can be found in [6]. However, in this thesis, we are interested in only the second order elliptic and parabolic equations which can be dealt through the use of Wasow's technique [10] and Levinson's technique [7].

Now in the case of the elliptic problem, the result obtained by Levinson [7] is of the form

$$u(x,y;\varepsilon) = U(x,y) + z(x,y;\varepsilon) + w(x,y;\varepsilon)$$

where $w = O(\sqrt{\varepsilon})$ uniformly over D. In this thesis, we show how this error or remainder term may be refined to $w = O(\varepsilon)$; and indeed present a method of obtaining a representation of u in D in which the remainder term in $O(\varepsilon^n)$ where n is any positive integer. This type of analysis is extended further to the case where the associated reduced equation

$$L_0[U] = a(x,y)U_x + b(x,y)U_y + c(x,y)U = f(x,y)$$

is replaced by a similar first order operator in which a, b, c, f as well as the boundary conditions are expressed as polynomials in ε.

The discussion of the parabolic problem [1] gives rise to results that are of the same nature as in the elliptic problem [7] (except that in this case Aronson had already developed a method for

showing the remainder term to be $O(\varepsilon)$, which in this thesis is applied to the elliptic case). Once again a method is devised of improving the accuracy of the representation from containing an error term which is $O(\varepsilon)$ to one which is $O(\varepsilon^n)$ as $\varepsilon \to 0^+$, and this type of analysis extended to the case where a, b, c, f and the boundary conditions are expressed as polynomials in ε.

The conditions under which the results of the elliptic problem which are obtained in [7] are rather restrictive, and so in Chapter IV we present a paper by Wasow [10] in which a manner of relaxing these restrictions is given in a special case. In this presentation, we append a comment on the manner in which the sign of ε as $\varepsilon \to 0$ determines the portion of the boundary over which the boundary layer phenomenon will be exhibited. We also show how the method of analysis developed by Wasow is applicable to the case of the corresponding parabolic problem.

CHAPTER II

LEVINSON'S METHOD FOR DEALING WITH THE ELLIPTIC SECOND ORDER PDE $\varepsilon\Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y)$

1. Introduction

In 1950, Levinson in [7] studied the asymptotic expansion as $\varepsilon \to 0^+$ of $u(x,y;\varepsilon)$, the solution of the first boundary value problem for

$$\varepsilon\Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \tag{1.1}$$

for small ε, under certain mild conditions in a domain D. The main tools of the proof are the maximum principle for elliptic differential equations, the theory of characteristics of first order partial differential equations, and an integral inequality to prove the smallness of the remainder. His error or remainder estimate, however, is not the best possible, for we can improve it from $O(\sqrt{\varepsilon})$ to $O(\varepsilon)$ by means of constructing a suitable comparison function as shown in §7. Furthermore, in §§8,9, by obtaining more terms in the formal expansion (5.2) in §5 and (6.1) in §6 we show how to obtain the solution with an error term equal to $O(\varepsilon^{N+1})$ for any given integer $N > 0$. We also study the case where the coefficients of the elliptic equation can be expanded as polynomials in ε.

2. Statement of the Problem

Let D be an open simply or multiply connected region in

the xy-plane. Γ_o is a portion of the boundary of D which is nowhere parallel to the characteristics of problem P_1 below. We consider problems

$$P_1: \quad \text{PDE} \quad a(x,y)U_x + b(x,y)U_y + c(x,y)U = d(x,y) \quad (x,y) \in D \qquad (2.1)$$

$$\text{IC} \qquad U = \phi \qquad (x,y) \in \Gamma_o$$

$$P_2: \quad \text{PDE} \quad \varepsilon\Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \quad (x,y) \in D \qquad (2.2)$$

$$\text{BC} \qquad u = \begin{cases} \phi & (x,y) \in \Gamma_o \\ \psi & (x,y) \in (\partial D - \Gamma_o) \end{cases}$$

where ψ is chosen such that $u \in C(\partial D)$. We are interested in the behavior of $u(x,y;\varepsilon)$ as $\varepsilon \to 0^+$ and, in particular, in its relation to $U(x,y)$ of P_1.

The characteristics curves[(1)] of problem P_1 are the solution of

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} \qquad (2.3)$$

First we require the following hypotheses:

<u>H_1</u>. Let ∂D, the boundary of D, consist of a finite number of simply closed curves ∂D_i $(i = 1,2,\ldots,n)$. Thus $D+\partial D$ is a closed connected set. Let $D+\partial D$ be contained in an open connected region

---

(1) See [3], pp. 62-64.

D_o . Assume $a(x,y)$, $b(x,y)$, $c(x,y)$ and $d(x,y)$ in $C^6(D_o)$. ∂D_i $(i = 1,2,\ldots,n)$ are given by functions yielding x and y in terms of arc length along each closed curve; let these functions and their partial derivatives up to sixth order with respect to arc length be continuous. Let the boundary values assigned to u along each closed curve ∂D_i of D and their partial derivatives up to sixth order with respect to arc length be continuous.

<u>H_2</u>. There exists a function $\Lambda(x,y) \in C^2(D_o)$ such that

$$a\Lambda_x + b\Lambda_y > 0 \quad (x,y) \in D_o \quad .$$

<u>Remark 2.1</u>. H_2 implies $a^2 + b^2 > 0$ for $(x,y) \in D$, since if $a^2 + b^2 = 0$ at some point, then $a = b = 0$ so that $a\Lambda_x + b\Lambda_y = 0$ too.

<u>H_2 alternate</u>. $c(x,y) < 0 \quad (x,y) \in D_o$.

<u>Remark 2.2</u>. Throughout we may replace H_2 by H_2 alternate, as theorem 3.2 shows.

3. <u>Maximum Principle</u>

<u>Theorem 3.1</u>. Let $u(x,y) \in C^2(D)$ satisfy

$$\Delta u + \alpha(x,y)u_x + \beta(x,y)u_y + \gamma(x,y)u = 0$$

where $\alpha,\beta,\gamma \in C(D)$, $\gamma < 0$ for $(x,y) \in D$ and D is an open domain. Then u cannot have a positive maximum nor a negative minimum in D .

Corollary. If in D

$$\Delta u + \alpha(x,y)u_x + \beta(x,y)u_y + \gamma(x,y)u = \rho(x,y)$$

and $\gamma \leq -K < 0$, $|\rho| \leq M$ for $(x,y) \in D$, then u cannot have a postive maximum nor negative minimum in D exceeding M/K in magnitude. If $u \in C(D+\partial D)$ and $\max_{(x,y)\in\partial D} |u(x,y)| \leq M/K$, then $\max_{(x,y)\in D} |u(x,y)| \leq M/K$.

Proof. If u has a positive maximum at a point $p \in D$, then $\Delta u(p) \leq 0$ and $u_x(p) = u_y(p) = 0$. Thus

$$(\Delta u + \alpha u_x + \beta u_y + \gamma u)\Big|_p = \rho(p)$$

$$\gamma(p) \,.\, u(p) = \rho(p) - \Delta u(p) \geq \rho(p) \quad .$$

Hence

$$u(p) \leq \frac{\rho(p)}{-\gamma(p)} \leq M/K \quad .$$

Similarly, if u has a negative minimum at the point p , then

$$\gamma(p) \cdot u(p) = \rho(p) - \Delta u(p)$$

$$\leq \rho(p) \quad .$$

This implies that

$$u(p) \geq \frac{\rho(p)}{-\gamma(p)} \geq - M/k$$

Theorem 3.2. Given H_1 and H_2 and $u(x,y) \in C^2(D)$ satisfies

$$\varepsilon \Delta u + a u_x + b u_y + c u = d \quad .$$

If there exists a constant m such that

$$\max_{(x,y)\varepsilon(D+\partial D)} |d(x,y)| \leq m \quad \text{and} \quad \max_{(x,y)\varepsilon\partial D} |u(x,y)| \leq m \quad ,$$

then there exists a constant K independent of ε for small $\varepsilon > 0$ such that

$$|u(x,y)| < Km \qquad (x,y) \ \varepsilon \ D \quad .$$

<u>Proof</u>. Let $v(x,y) = e^{A\Lambda(x,y)} \cdot u(x,y)$ where $A > 0$ is a constant such that

$$A(a\Lambda_x + b\Lambda_y) > 0 \quad .$$

By H_2 , there exists such an A . Then

$$u = e^{-A\Lambda} \cdot v \quad ,$$

so that v satisfies

$$\varepsilon \Delta v + (a - 2\varepsilon A\Lambda_x) v_x + (b - 2\varepsilon A\Lambda_y) v_y$$

$$+ [c - A(a\Lambda_x + b\Lambda_y) + \varepsilon A^2(\Lambda_x^2 + \Lambda_y^2) - \varepsilon A \Delta\Lambda] v = d \cdot e^{A\Lambda} \quad .$$

Since all the functions in the coefficient of v are constant, we have that for a sufficiently small ε the coefficient of v is always negative so that we may apply the corollary in this section to get $|v(x,y)|$ is bounded in D . Since $u = v\ e^{-A\Lambda}$, and as Λ is a continuous function on a closed bounded domain, $e^{-A\Lambda}$ is bounded above so that $|u(x,y)|$ is also bounded above.

4. Some Conventions, An Existence and Uniqueness Theorem

Let us specify the positive direction on ∂D as the one taking you around counter-clockwise and denote the arc length on ∂D by s. Let Γ_o be an arc of one of the curves of ∂D and let no characteristic of (2.3) be tangent to Γ_o. Let Γ_1 be another arc of ∂D such that the characteristics through Γ_o into D intersect Γ_1 and so that Γ_1 also nowhere tangent to the characteristic curves of (2.3). Then the closed region bounded by Γ_o, Γ_1 and the two characteristics through the end points of Γ_o and Γ_1 is called a "regular quadrilateral".

Since neither Γ_o nor Γ_1 has any characteristic of (2.3) tangent to it, we have on ∂D

$$b \frac{dx}{ds} - a \frac{dy}{ds} \neq 0 \quad .$$

If on Γ_o we have

$$b \frac{dx}{ds} - a \frac{dy}{ds} < 0 \tag{4.1}$$

then on Γ_1 we have

$$b \frac{dx}{ds} - a \frac{dy}{ds} > 0 \quad . \tag{4.2}$$

Theorem 4.1. Given H_1 and H_2 and problem P_2 of §2 . Then there exists a solution $u(x,y;\varepsilon)$ for any small $\varepsilon > 0$ and $u,u_x,u_y \; \varepsilon \; C(D+\partial D)$, $u \; \varepsilon \; C^2(D)$.

Theorem 4.1 follows from results of Lichtenstein [8].

Remark. For $\varepsilon < 0$, the roles of Γ_o and Γ_1 are interchanged. For a proof see §2.2, chapter IV.

5. Statement and Proof of the Main Theorem in a Special Case

Define the problem

$$P_3: \quad \text{PDE} \quad \varepsilon\Delta u + au_x + bu_y + cu = 0 \quad (x,y) \in D \tag{5.1}$$

$$\text{BC} \quad u = \begin{cases} \theta(s) & (x,y) \in \Gamma_1 \\ 0 & (x,y) \in (\partial D - \Gamma_1) \end{cases}$$

whose solution is $u(x,y;\varepsilon)$ and we assumed that $\theta(s) \in C^2(\Gamma_1)$ where s is arc length along Γ_1 and $\theta(s) \equiv 0$ over a small segment at each end of Γ_1. The solution of the associated reduced problem is, of course, $U(x,y) \equiv 0$.

Main Theorem. Given H_1 and H_2 (or H_2 alternate) and problem P_3. Then in any regular quadrilateral Q with sides Γ_o and Γ_1, as above, in $D+\partial D$ we have

$$u(x,y;\varepsilon) = z(x,y;\varepsilon) + \varepsilon Z(x,y;\varepsilon) \tag{5.2}$$

where

$$Z(x,y;\varepsilon) = \begin{cases} O(1) & \text{uniformly in } Q \text{ as } \varepsilon \to 0^+ \\ 0 & \text{on } \partial D \end{cases}$$

and

$$z(x,y;\varepsilon) = e^{\frac{-g(x,y)}{\varepsilon}} \cdot h(x,y) \quad \text{on } \Gamma_1 \text{ and at points in } Q \text{ near } \Gamma_1 \tag{5.3}$$

where (i) $g,h \in C^2(Q)$, (ii) $g = 0$, $h = \theta(s)$ on Γ_1 , and (iii) $g > 0$ inside Q . There exists $\delta > 0$ such that $z = O(e^{-\delta/\varepsilon})$ as $\varepsilon \to 0^+$ uniformly in that part of Q where (5.3) no longer holds.

<u>Proof</u>. By corollary, §3 in the case of H_2 alternate or by theorem 3.2 in the case of H_2 , $u(x,y;\varepsilon)$ is uniformly bounded in $D+\partial D$ as $\varepsilon \to 0^+$. If we substitute (5.2) into (5.1) we obtain

$$\begin{aligned}&\frac{h}{\varepsilon}[g_x^2 + g_y^2 - (ag_x + bg_y)]\, e^{-(g/\varepsilon)} \\ &\quad + [(a - 2g_x)h_x + (b - 2g_y)h_y + (c - \Delta g)h]e^{-(g/\varepsilon)} \\ &\quad + \varepsilon\Delta h e^{-(g/\varepsilon)} + \varepsilon[\varepsilon\Delta Z + aZ_x + bZ_y + c] = 0 .\end{aligned}$$

We obtain $g(x,y)$ by setting the coefficient of $\frac{h}{\varepsilon} e^{-(g/\varepsilon)}$ equal to zero to get the following problem for $g(x,y)$:

P_4 : PDE $g_x^2 + g_y^2 - (ag_x + bg_y) = 0$ in Q near Γ_1

IC $g = 0$ on Γ_1 and $g > 0$ in Q near Γ_1 .

We obtain $h(x,y)$ by setting the coefficient of $e^{-(g/\varepsilon)}$ equal to zero to get the following problem for $h(x,y)$:

P_5 : PDE $(a - 2g_x)h_x + (b - 2g_x)h_y + (c - \Delta g)h = 0$ in Q near Γ_1

IC $h = \theta(s)$ on Γ_1 .

Now we solve problem P_4 . Let $p = g_x$, $q = g_y$. Then the

characteristic system for P_5 is

$$\frac{dx}{dt} = 2p - a \ , \qquad \frac{dy}{dt} = 2q - b \ ,$$

$$\frac{dp}{dt} = a_x p + b_x q \ , \qquad \frac{dq}{dt} = a_y p + b_y q \ , \tag{5.4}$$

$$\frac{dg}{dt} = p^2 + q^2 \ , \qquad p^2 + q^2 = ap + bq \ .$$

We shall show that for our purpose the characteristics of (2.3) determine the segment of ∂D over which we make use of (5.4) and that (4.2) is a sufficient condition for the existence of the solution of P_4 .

To get $g(x,y)$ near Γ_1 in Q , it is sufficient to show that no characteristic is tangent to Γ_1 to show $g > 0$ in Q near Γ_1 , observe from (5.4) that $\frac{dg}{dt} = p^2 + q^2$ will suffice provided $p = 0$ and $q = 0$ do not simultaneously hold and $dt > 0$ as we go into Q .

Now on Γ_1 we have, as $g = 0$, that

$$\frac{\partial g}{\partial s} = px_s + qy_s = 0 \ .$$

From (5.4) we have

$$p^2 + q^2 = ap + bq \ .$$

Thus on Γ_1 we may solve to get

$$q = -p \cdot \frac{x_s}{y_s} \ .$$

Hence

$$p = (ay_s - bx_s)y_s$$
$$q = -(ay_s - bx_s)x_s \tag{5.5}$$

(as $x_s^2 + y_s^2 \equiv 1$, and we reject the solution $p \equiv q \equiv 0$). By (5.4), (5.5) and (4.2) we obtain

$$x_s y_t - x_t y_s = bx_s - ay_s > 0 \quad \text{on} \quad \Gamma_1 \quad .$$

Thus no characteristic of (5.4) is tangent to Γ_1 . Further (4.2) implies that the characteristics of (5.4) go into Q as t increases from 0 . Again (5.5) implies $p^2 + q^2 = (ay_s - bx_s)^2 > 0$ on Γ_1 . Thus $\frac{dg}{dt} > 0$ on Γ_1 and, by continuity, in Q near Γ_1 . Hence, as $g = 0$ on Γ_1 , $g > 0$ in Q near Γ_1 .

Let the arc Γ_1 , ABCD , have A and D as its end points. Γ_1 is given parametrically as $x = x(s)$, $y = y(s)$. $\theta(s) = 0$ on AB and CD . The initial values at $t = 0$ on Γ_1 for (5.4) read:

$$x(s,0) = x(s) \quad , \quad y(s,0) = y(s)$$

$$g(s,0) = 0 \ ,$$

$$p(s,0) = -[b(x(s),y(s) \cdot x_s(s) - ay_s]y_s(s) \ ,$$

$$q(s,0) = [bx_s - ay_s] \cdot x_s \quad .$$

Since $\Delta(s,t) = \frac{\partial(x,y)}{\partial(s,t)} = bx_s - ay_s > 0$ on Γ_1 , then by continuity $\Delta(s,t) > 0$ for some region extending into Q . As a, b, x(s), y(s)

are all of class C^6 by H_1 , we have that $g \in C^4$.[1]

Let arcs BF and CE be the projections of the characteristics of (5.4) into the xy-plane. T_1 is the closed region ABCDEFA , and AFED is sufficiently close to Γ_1 that T_1 lies in the open set in which g exists. Also AFED is chosen such that

(i) x,y along AFED expressed in terms of arc length are of class C^3 ,

(ii) the normals to AFED outward to T_1 are directed into Q (this is arranged by making $\angle$ BAF and $\angle$ CDE acute),

(iii) AFED is sufficiently close to Γ_1 that $p^2 + q^2 > 0$ in T_1 (this is possible as $p^2 + q^2 > 0$ on Γ_1) (see figure 2.1).

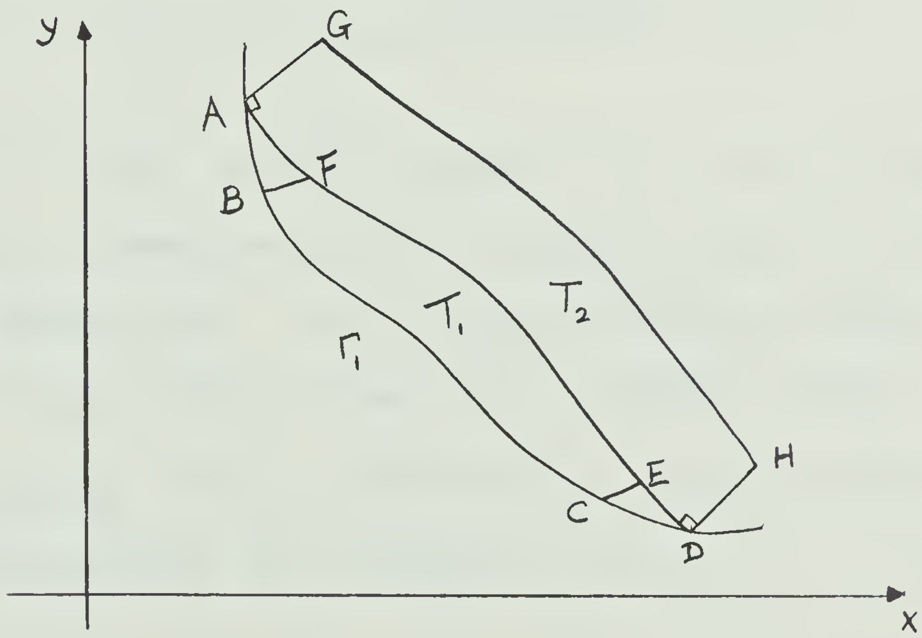

Figure 2.1

---

[1] See [2], theorems 7.1 and 7.2, pp. 24-25.

Since $\frac{dg}{dt} = p^2 + q^2 > 0$, there exists $\delta > 0$ such that $g(x,y) > \delta t$ in T_1 and there exists $\delta_1 > 0$ such that $g(x,y) \geq 2\delta_1$ for (x,y) on arc EF .

We now solve problem P_5 . The characteristic system for P_5 is

$$\frac{dx}{dt} = -a + 2g_x \ , \quad \frac{dy}{dt} = -b + 2g_y$$
$$\frac{dh}{dt} = (c-\Delta g)h \tag{5.6}$$

where $g_x = p$ and $g_y = q$. Since the first two equations of (5.6) and the first two equations of (5.4) are the same, the characteristics of P_4 and P_5 in the xy-plane are the same.

The last equation of (5.6) implies that

$$h(s,t) = h(s)\, e^{\int_o^t (c-\Delta g)dt} \ .$$

Since $g \in C^4$, then $\Delta g \in C^2$. Hence $h \in C^3$ for t , but $h \in C^2$ for s . Therefore $h \in C^2$. $h(s,0) = \theta(s) = 0$ for s on AB and CD implies that $h = 0$ in the curvilinear triangles ABF and CDE . As $g,h \in C^2(T_1)$ then $z = h \cdot e^{-(g/\varepsilon)} \in C^2(T_1)$. z, z_x, z_y, z_{xx}, z_{xy} and z_{yy} vanish on AF and ED and are all $O(e^{-\delta_1/\varepsilon})$ uniformly on EF for small $\varepsilon > 0$.

We now extend the definition of z a little further into Q . Let σ be the arc length along AFED so that AFED is given parametrically by $x = x(\sigma)$, $y = y(\sigma)$. We erect a family of normals to AFED exterior to T_1 with τ as arc length on the normals.

These normals are given by

$$x(\tau,\sigma) = x(\sigma) - y'(\sigma) \cdot \tau \quad ,$$

$$y(\tau,\sigma) = y(\sigma) + x'(\sigma) \cdot \tau \quad .$$

Since $\frac{\partial(x,y)}{\partial(\sigma,\tau)} = 1 + (x''y' - x'y'')\tau$, on $\tau = 0$ (i.e. AFED) , we have that $\frac{\partial(x,y)}{\partial(\sigma,\tau)} = 1 \neq 0$. So that there exists a 1-1 correspondence between (x,y) and (σ,τ) near $\tau = 0$. Now choose τ_1 such that $\frac{\partial(x,y)}{\partial(\sigma,\tau)} \neq 0$ for $0 \leq \tau \leq \tau_1$. Let GH be the curve along which we have $\tau = \tau_1$. Let T_2 be the curvilinear quadrilateral AFEDHGA (Fig. 2.1). We extend $z(x,y;\varepsilon)$ into T_2 such that $z = 0$ in $(Q - T_1 - T_2)$ and $z \in C^2(Q)$ as follows.

Define

$$z = (1 - \frac{\tau}{\tau_1})^3 \, [z_1(\sigma) + \tau \cdot z_2(\sigma) + \tau^2 \cdot z_3(\sigma)] \qquad (5.7)$$

where $z_1(\sigma) = z$, $z_2(\sigma) = \frac{\partial z}{\partial \tau} + \frac{3}{\tau_1} z_1(\sigma)$ and $z_3(\sigma) = \frac{1}{2}\frac{\partial^2 z}{\partial \tau^2} + \frac{3}{\tau_1} z_2(\sigma) - \frac{3}{\tau_1^2} z_1(\sigma)$ on $\tau = 0$. The functions z, z_x, z_y, z_{xx}, z_{xy}, z_{yy} all vanish on the arc GH , since all involve a factor $(1 - \frac{\tau}{\tau_1})$. Due to the fact that $z \equiv 0$ in the curvilinear triangles ABF and CDE , z and its partial derivatives up to second order vanish on AF and DE and are $O(e^{-\delta_1/\varepsilon})$ for $(x,y) \in (Q-T_1)$.

Recall that by (5.2) we have

$$u(x,y;\varepsilon) = z(x,y;\varepsilon) + \varepsilon Z(z,y;\varepsilon) \quad .$$

By (5.3), $z = e^{-(g/\varepsilon)} \cdot h$ in T_1 where g and h are solutions of P_4 and P_5, respectively. By substituting into (5.1) we obtain

$$\varepsilon \Delta Z + aZ_x + bZ_y + cZ = -h \cdot e^{-(g/\varepsilon)} \qquad (x,y) \in T_1 \quad . \tag{5.8}$$

Again z is given by (5.7) in T_2 so that we have

$$\begin{aligned} \varepsilon \Delta Z + aZ_x + bZ_y + cZ &= \frac{1}{\varepsilon}(\varepsilon \Delta z + az_x + bz_y + cz) \\ &= O(\frac{1}{\varepsilon} e^{-\delta_1/\varepsilon}) \\ &\text{as } \varepsilon \to 0^+ \quad (x,y) \in T_2 \quad . \end{aligned} \tag{5.9}$$

Lastly, $z = 0$ in $(D+\partial D-T_1-T_2)$ so that we have

$$\varepsilon \Delta Z + aZ_x + bZ_y + cZ = 0 \qquad (x,y) \in (Q-T_1-T_2) \quad . \tag{5.10}$$

Thus from (5.8), (5.9) and (5.10) we have that Z solves the problem

$$\begin{aligned} &\text{PDE} \quad \varepsilon \Delta Z + aZ_x + bZ_y + cZ = O(1) \quad \text{uniformly} \quad (x,y) \in D+\partial D, \quad \varepsilon > 0 \\ &\text{BC} \quad Z = 0 \qquad\qquad (x,y) \in \partial D \quad . \end{aligned}$$

Applying the maximum principle of theorem 3.2 to this problem we have that

$$Z(x,y;\varepsilon) = O(1) \tag{5.11}$$

uniformly in $D+\partial D$ as $\varepsilon \to 0^+$. Hence

$$u(x,y;\varepsilon) = z(x,y;\varepsilon) + O(\varepsilon)$$

uniformly in $D+\partial D$ as $\varepsilon \to 0^+$.

6. Statement and Proof of the Main Theorem in the General Case

In this section, instead of dealing with the special case of problem P_3 and its associated reduced problem we return to problems P_1 and P_2 .

Main Theorem. Given H_1, H_2 (or H_2 alternate), problems P_1 and P_2 with solutions $U(x,y)$ and $u(x,y;\varepsilon)$, respectively. Then in any regular quadrilateral Q with sides Γ_o and Γ_1 as given by (4.1) and (4.2) in $D+\partial D$ we have that

$$u(x,y;\varepsilon) = U(x,y) + z(x,y;\varepsilon) + \varepsilon Z(x,y;\varepsilon) + w(x,y;\varepsilon) \tag{6.1}$$

where $w = O(\sqrt{\varepsilon})$ as $\varepsilon \to 0^+$ uniformly in Q and $w = 0$ on Γ_o and Γ_1 .

$$z = e^{\frac{-g(x,y)}{\varepsilon}} \cdot h(x,y) \tag{6.2}$$

on Γ_1 and at points in Q near Γ_1 , where $g,h \in C^2(Q)$; $g = 0$ on Γ_1 and $g > 0$ inside Q near Γ_1 ; $h = \psi - U$ on Γ_1 . $Z = O(1)$ uniformly in Q as $\varepsilon \to 0^+$ and $Z = 0$ on ∂Q . There exists $\delta > 0$ such that $z = O(e^{-\delta/\varepsilon})$ uniformly in that part of Q where (6.2) no longer holds.

Proof. By characteristics in this section we mean the characteristics of (2.3) and not of (5.4). The characteristics of (2.3) can be designated by the coordinate of arc length s on Γ_1 . Let s at the end points

of Γ_1 be given by s_1 and s_2. Since (4.1) and (4.2) are inequalities, we can always extend Γ_o and Γ_1 thereby enlarging Q such that it contains the characteristics given

$$s_1 - 3\delta \leq s \leq s_2 + 3\delta \qquad \text{for some } \delta > 0 \ .$$

Denote the regular quadrilateral given by $s_1 - 2\delta \leq s \leq s_2 + 2\delta$ by $\overline{Q}$ and the segments of $\overline{Q}$ by $\overline{\Gamma}_o$ and $\overline{\Gamma}_1$. Let

$$\theta(s) = \begin{cases} \psi - U & \text{on } \overline{\Gamma}_1 \\ 0 & \text{on } \overline{\Gamma}_1 \text{ near } s_1 - 3\delta \text{ and } s_2 + 3\delta \end{cases}$$

Let $\theta(s) \varepsilon C^2$ on $[s_1-3\delta, s_2+3\delta]$. Let $z(x,y;\varepsilon)$ and $Z(x,y;\varepsilon)$ be given as in §5, and $w(x,y;\varepsilon)$ be given by (6.1). $w(x,y)$ is uniformly bounded in $\overline{Q}$ as $\varepsilon \to 0^+$, since $u(x,y;\varepsilon)$, $U(x,y)$, $z(x,y;\varepsilon)$ and $Z(x,y;\varepsilon)$ are uniformly bounded in $\overline{Q}$ as $\varepsilon \to 0^+$. Now $U(x,y)$ satisfies

$$\begin{array}{lll} \text{PDE} & aU_x + bU_y + cU = d & (x,y) \ \varepsilon \ \overline{Q} \\ \text{IC} & U = \phi & (x,y) \ \varepsilon \ \overline{\Gamma}_o \end{array}$$

And $u(x,y;\varepsilon)$ satisfies problem P_2. Also z is given by

$$z = \begin{cases} e^{-(g/\varepsilon)} \cdot h & (x,y) \ \varepsilon \ T_1 \\ (6.7) & (x,y) \ \varepsilon \ T_2 \\ 0 & (x,y) \ \varepsilon \ (Q-T_1-T_2) \end{cases}$$

where g,h solve P_4 and P_5 of §5, respectively. Lastly the problem

Z solves is given in §5 . Hence w satisfies the problem:

$$P_6 : \quad \text{PDE} \quad \varepsilon\Delta w + aw_x + bw_y + cw = -\varepsilon\Delta U \quad (x,y) \in \overline{Q} \tag{6.3}$$

$$\text{BC} \quad w = 0 \quad (x,y) \in (\overline{\Gamma_o} + \overline{\Gamma_1}) \; .$$

w in general is non-zero on $\partial D - (\overline{\Gamma_o} + \overline{\Gamma_1})$; indeed $w = u$ there except for a small segment at each end. w is of class $C^2(\overline{Q})$, since $z \in C^2(\overline{Q})$.

Instead of using the characteristic integral inequality of Levinson [7] to obtain that $w = O(\sqrt{\varepsilon})$ in Q , we shall show that in next section, $w = O(\varepsilon)$ in Q by means of a suitable comparison function.

7. The Estimate of $w(x,y)$

Let $Q = Q_o$ be embeded in a series of regular quadrilaterals Q_k $(k = 0,1,2,3)$ which are bounded by the characteristics $s = s_1 - k\delta$ and $s = s_2 + k\delta$, and the sides $\Gamma_k^{(i)}$ $(i = 1,2)$. Let $\sigma = \sigma_k^{(i)}$ $(i = 1,2)$ be the characteristics bounding Q_k $(k = 0,1,2)$ (see figure 2.2).

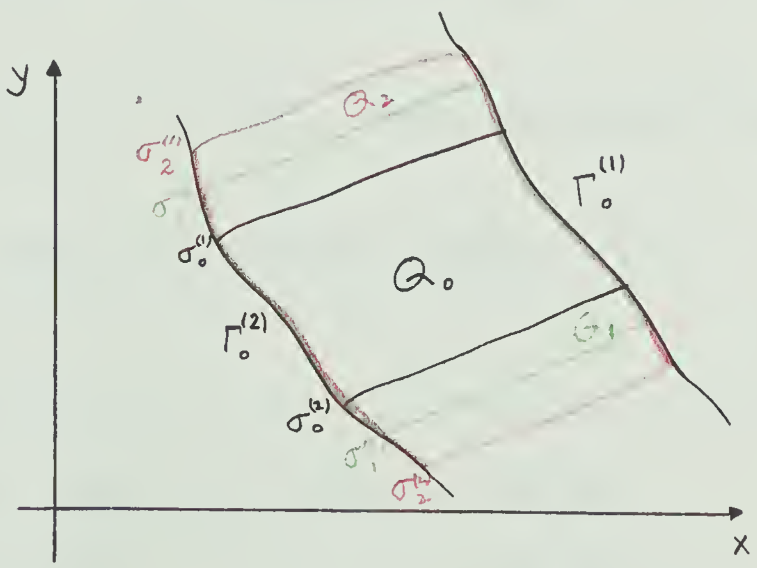

Figure 2.2

Lemma 7.1. Let H_1 and H_2 (or H_2 alternate) hold. If $W = W(x,y;\varepsilon)$ is the solution of

$$\text{PDE} \quad \varepsilon\Delta W + aW_x + bW_y + cW = 0 \quad \text{in} \quad Q_k \tag{7.1}$$

$$\text{BC} \quad \begin{cases} W = 0 & \text{on} \quad \Gamma_k^{(1)} \quad \text{and} \quad \Gamma_k^{(2)} \\ |W| < K\,\varepsilon^n & \text{on} \quad \sigma = \sigma_k^{(1)} \quad \text{and} \quad \sigma = \sigma_k^{(2)} \end{cases} \tag{7.2}$$

where K is a constant independent of ε, then

$$W = 0(\varepsilon^{n+\frac{1}{2}}) \quad \text{uninformly in} \quad Q_{k-1} \quad \text{as} \quad \varepsilon \to 0^+ . \tag{7.4}$$

Proof. Suppose $V = V(x,y;\varepsilon)$ solves the problem

$$\text{PDE} \quad \varepsilon\Delta V + aV_x + bV_y + cV = 0 \quad \text{in} \quad Q_k \tag{7.5}$$

$$\text{BC} \quad \begin{cases} V \geq 0 & \text{on} \quad \Gamma_k^{(1)} \quad \text{and} \quad \Gamma_k^{(2)} & (7.6) \\ V \geq K\,\varepsilon^n & \text{on} \quad \sigma = \sigma_k^{(1)} \quad \text{and} \quad \sigma = \sigma_k^{(2)} & (7.7) \end{cases}$$

$$\text{with} \quad V = 0(\varepsilon^{n+\frac{1}{2}}) \quad \text{uniformly in} \quad Q_{k-1} \quad \text{as} \quad \varepsilon \to 0^+ . \tag{7.8}$$

By theorem 3.1, $V \geq 0$ in Q_k. Now consider the function

$$P(x,y;\varepsilon) = V(x,y;\varepsilon) - W(x,y;\varepsilon) \quad .$$

Then P solves

$$\text{PDE} \quad \varepsilon\Delta P + aP_x + bP_y + cP = 0 \quad \text{in} \quad Q_k$$

$$\text{BC} \quad P \geq 0 \quad \text{on} \quad \partial Q_k \quad .$$

By theorem 3.1, $P \geq 0$ uniformly in Q_k. Therefore $V \geq W$ uniformly in Q_k. Similarly, by considering $P = V + W$, we find that $V + W \geq 0$ or $W \geq -V$ uniformly in Q_k so that $|W| \leq |V|$. Provided we can exhibit a V such that (7.5), (7.6), (7.7) and (7.8) hold, then

$$W = O(\varepsilon^{n+\frac{1}{2}}) \quad \text{uniformly in } Q_k .$$

To find such a V, consider the function

$$V(x,y;\varepsilon) = \sum_{\alpha=1}^{2} e^{-\gamma_k^{(\alpha)}} \cdot h^{(\alpha)}(x,y;\varepsilon) + \sqrt{\varepsilon}\, Y(x,y;\varepsilon) ,$$

where $\gamma_k^{(\alpha)} = \frac{1}{\sqrt{\varepsilon}} (\sigma-\sigma_k^{(\alpha)})^2$. Substitute into (7.5) to get

$$\sum_{\alpha=1}^{2} e^{-\gamma_k^{(\alpha)}} [ah_x^{(\alpha)} + bh_y^{(\alpha)} + \{c+4(\sigma-\sigma_k^{(\alpha)})(\sigma_x^2 + \sigma_y^2)\}h^{(\alpha)}]$$

$$+ \sqrt{\varepsilon}G(x,y;\varepsilon) + \sqrt{\varepsilon}(\varepsilon\Delta Y + aY_x + bY_y + cY) = 0 , \tag{7.9}$$

where

$$G(x,y;\varepsilon) = \sum_{\alpha=1}^{2} e^{-\gamma_k^{(\alpha)}} [\sqrt{\varepsilon}\, \Delta h^{(\alpha)} -4(\sigma-\sigma_k^{(\alpha)})(\sigma_x h_x^{(\alpha)} + \sigma_y h_y^{(\alpha)})$$

$$-2\{(\sigma-\sigma_k^{(\alpha)})\Delta\sigma + (\sigma_x^2 + \sigma_y^2)\}h^{(\alpha)}] . \tag{7.10}$$

We take $h^{(\alpha)} = h^{(\alpha)}(x,y;\varepsilon)$ to be the solution of the initial value problem

$$\text{PDE} \quad ah_x^{(\alpha)} + bh_y^{(\alpha)} + [c+4(\sigma-\sigma_k^{(\alpha)})(\sigma_x^2 + \sigma_y^2)]h^{(\alpha)} = 0 \quad \text{in} \quad Q_k \tag{7.11}$$

$$\text{IC} \quad h^{(\alpha)} = \frac{K\,\varepsilon^n}{N} \quad \text{on} \quad \Gamma_k^{(1)}$$

where $N > 0$ is a constant specified in (7.13). Now the characteristic equations of (7.11) are

$$\frac{dx}{dt} = a \ , \quad \frac{dy}{dt} = b \ ,$$

$$\frac{dh^{(\alpha)}}{dt} = [c + 4(\sigma-\sigma_k^{(\alpha)})(\sigma_x^2 + \sigma_y^2)]h^{(\alpha)} \quad . \tag{7.12}$$

Hence the characteristics of (7.11) on the xy-plane are the same as those of the reduced equation (2.1). Hence $h^{(\alpha)}$ exists uniquely in Q_k and is in class $C^5(Q_k)$. By (7.12), and $t = 0$ on $\Gamma_k^{(1)}$,

$$h^{(\alpha)} = \frac{K}{N}\varepsilon^n \cdot e^{\int_o^t [c+4(\sigma-\sigma_k^{(\alpha)})(\sigma_x^2+\sigma_y^2)]dt} \quad .$$

On $\sigma = \sigma_k^{(\alpha)}$, $h^{(\alpha)} = \frac{K}{N} \varepsilon^n e^{\int_o^t c\,dt}$. Thus if we let

$$N = \min_{(\sigma,t)\,\varepsilon\, Q_k} e^{\int_o^t c\,dt} \tag{7.13}$$

then $0 < N \leq 1$ and $h^{(\alpha)} \geq K\,\varepsilon^n$ on $\sigma = \sigma_k^{(\alpha)}$.

From (7.10), $G \,\varepsilon\, C^3(Q_k)$ and $G = 0(\varepsilon^n)$ uniformly in Q_k . From (7.9), Y solves

$$\text{PDE} \quad \varepsilon\Delta Y + aY_x + bY_y + cY = -G \quad \text{in} \quad Q_k$$

$$\text{BC} \quad Y = 0 \quad \text{on} \quad \partial Q_k \ .$$

Now by theorem 3.1, $Y = O(\varepsilon^n)$ uniformly in Q_k for small enough $\varepsilon > 0$. Thus we have found the required V.

Theorem 7.1. Let H_1 and H_2 (or H_2 alternate) hold, and $w(x,y;\varepsilon)$ be the solution of

$$\text{PDE} \qquad \varepsilon\Delta w + aw_x + bw_y + cw = -\varepsilon\Delta U \qquad (x,y) \in Q_2$$

$$\text{BC} \qquad \begin{cases} w = 0 & \text{on } \Gamma_2^{(1)} \text{ and } \Gamma_2^{(2)} \\ |w| < B & \text{on } \sigma = \sigma_2^{(1)} \text{ and } \sigma = \sigma_2^{(2)} \end{cases}$$

where B is independent of ε. Then $w = O(\varepsilon)$ uniformly in Q_o as $\varepsilon \to 0^+$.

Proof. Since we are interested only in the order of magnitude of w, we can assume without loss of generality that $c(x,y) < 0$ in Q_2. For if it is not, we need merely use the substitution of theorem 3.2 to insure it.

Now let $W_o = W_o(x,y;\varepsilon)$ solve the problem

$$\text{PDE} \qquad \varepsilon\Delta W_o + aW_{o,x} + bW_{o,y} + cW_o = -\varepsilon\Delta U \qquad \text{in } Q_2$$

$$\text{BD} \qquad W_o = 0 \qquad \text{on } \partial Q_2 .$$

By theorem 3.2, $W_o = O(\varepsilon)$ uniformly in Q_2. Put $w = W_o + W$. Then W solves the problem

$$\text{PDE} \qquad \varepsilon\Delta W + aW_x + bW_y + cW = 0 \quad \text{in } Q_2$$

$$\text{BC} \quad \begin{cases} W = 0 & \text{on } \Gamma_2^{(1)} \text{ and } \Gamma_2^{(2)} \\ |W| < B & \text{on } \sigma = \sigma_2^{(1)} \text{ and } \sigma = \sigma_2^{(2)} \end{cases} .$$

Then by lemma 7.1 with $n = 0$ and $k = 2$ we have that $W = O(\sqrt{\varepsilon})$ uniformly in Q_1 , i.e. there exists $K > 0$, independent of ε such that $|W| < K\sqrt{\varepsilon}$ in Q_1 . By applying lemma 7.1 again with $n = 1/2$ and $k = 1$, it follows that $W = O(\varepsilon)$ uniformly in Q_o as $\varepsilon \to 0^+$.

8. A Refinement of the Special Case of §5

In section 5 we have a solution of problem P_3 consisting of a boundary layer term plus an error or remainder term εZ which is $O(\varepsilon)$ in $D+\partial D$ as $\varepsilon \to 0^+$. In this section we shall improve the order of the remainder from 1 to N+1 in ε as $\varepsilon \to 0^+$. The notation of §5 will be retained, except for slight modifications introduced at the appropriate place. In particular, hypothesis H_1 is replaced by hypothesis H_1' in which we have:

(i) $a,b,c,d \in C^{2N+6}(D_o)$

(ii) The functions yielding x and y in terms of arc length specifying each closed curve are now of class C^{2N+6} in arc length.

(iii) $\theta(s) \in C^{2N+2}(\Gamma_1)$ and the boundary condition for u is of class $C^{2N+2}(\partial D)$.

Theorem 8.1. Given H_1', H_2 (or H_2 alternate) and problem P_3 with $\theta(s)$ modified as indicated in hypothesis H_1'. Then in any regular quaduilateral Q, with sides Γ_o and Γ_1 as before, in $D+\partial D$ we have that

$$u(x,y;\varepsilon) = z(x,y;\varepsilon) + \varepsilon^{N+1} Z(x,y;\varepsilon) \tag{8.1}$$

where $Z = 0(1)$ uniformly in Q as $\varepsilon \to 0^+$ and Z vanishes on ∂D, and

$$z = e^{-\frac{g(x,y)}{\varepsilon}} \sum_{n=o}^{N} \varepsilon^n h_n(x,y) \tag{8.2}$$

on Γ_1 and at points in Q near Γ_1, where

(a) $g \varepsilon C^{5+2N}(Q)$, $h_n \varepsilon C^{3+2(N-n)}(Q)$ $(n = 1,2,\ldots,N)$

(b) $g = 0$, $h_o = \theta$, $h_n = 0$ $(n = 1,\ldots,N)$ on Γ_1,

(c) $g > 0$ inside Q.

$\exists\ \delta > 0$, such that $z = 0(e^{-\delta/\varepsilon})$ as $\varepsilon \to 0^+$ uniformly in that part of Q where (8.2) no longer holds.

Proof. In the region T_1 we have z now given by (8.2). Thus the substitution of (8.1) into (5.1) will yield

$$\frac{1}{\varepsilon} e^{-(g/\varepsilon)}[(g_x^2+g_y^2)-(ag_x+bg_y)] \sum_{n=o}^{N} \varepsilon^n h_n$$

$$+ e^{-(g/\varepsilon)} \sum_{n=o}^{N} \varepsilon^n[(a-2g_x)\frac{\partial h_n}{\partial x} + (b-2g_y)\frac{\partial h_n}{\partial y} + (c-\Delta g)h_n]$$

$$+ e^{-(g/\varepsilon)} \sum_{n=o}^{N} \varepsilon^{n+1} \Delta h_n + \varepsilon^{N+1}[\varepsilon\Delta Z+aZ_x+bZ_y+cZ] = 0 \quad .$$

As before, we obtain $g(x,y)$ as the solution to problem P_4 and $h_o(x,y)$, which is the same as $h(x,y)$, as the solution to problem P_5 in §5 . For $h_n(x,y)$ we examine the coefficient of ε^n to obtain problem

$$P'_n \quad : \qquad \text{PDE} \quad (a-2g_x)\frac{\partial h_n}{\partial x} + (b-2g_y) + (c-\Delta g)h_n = -\Delta h_{n-1} \quad \text{in } Q \text{ near } \Gamma_1$$

$$\text{IC} \qquad h_n = 0 \qquad \text{on } \Gamma_1 \ .$$

Now as $a, b, x(s), y(s) \in C^{6+2N}$, we have that $g \in C^{5+2N}$.[1]

The characteristic LaGrange systems for problems P_5 and P'_n are respectively

$$\left\{\begin{aligned} \frac{dx}{dt} &= 2g_x - a \\ \frac{dy}{dt} &= 2g_y - b \\ \frac{dh_o}{dt} &= (c-\Delta g)h_o \end{aligned}\right. \qquad\qquad \left\{\begin{aligned} \frac{dx}{dt} &= 2g_x - a \\ \frac{dy}{dt} &= 2g_y - b \\ \frac{dh_n}{dt} &= (c-\Delta g)h_n + \Delta h_{n-1} \ . \end{aligned}\right.$$

Hence the characteristics in the xy-plane are the same for all these problems. Since

$$\frac{dh_o}{dt} = (c-\Delta g)h_o$$

and $g \in C^{5+2N}$, then $\Delta g \in C^{3+2N}$, so that $h_o \in C^{4+2N}$ in t and $h_o \in C^{3+2N}$ in s . A similar argument applies to h_n . Thus $h_n \in C^{3+2(N-n)}$.

---

[1] See [2] pp. 24-25.

Now

$$h_o(t,s) = h_o(0,s) \cdot e^{\int_o^t (c-\Delta g)dt} .$$

But $h_o(0,s) = \theta(s) = 0$ for s on AB and CD . Thus $h_o = 0$ on the curvilinear triangles ABF and CDE . Again the problem

$$\text{ODE} \qquad \frac{dh_n}{dt} = (c-\Delta g)h_n + \Delta h_{n-1}$$

$$\text{IC} \qquad h_n \Big|_{t=o} = 0$$

has solution

$$h_n(t,s) = e^{\int_o^t (c-\Delta g)dt} \int_o^t e^{\int_o^t (c-\Delta g)dt} \Delta h_{n-1}(t,s)dt .$$

But as $\Delta h_{n-1} \equiv 0$ in the curvilinear triangles ABF and CDE , so is $h_n \equiv 0$ there.

The extension of z into T_2 in §5 remains essentially the same. However the estimation of Z is changed to the following. Recall that z is defined as follows:

$$z = \begin{cases} e^{-(g/\varepsilon)} \sum_{n=o}^{N} \varepsilon^n h_n & (x,y) \in T_1 \\ (1 - \frac{\tau}{\tau_1})^3 \sum_{n=o}^{3} \tau \cdot z_n(\sigma) & (x,y) \in T_2 \\ 0 & (x,y) \in (D+\partial D-T_1-T_2) \end{cases} .$$

Thus the PDE for Z is given by

$$\varepsilon\Delta Z+aZ_x+bZ_y+cZ = \begin{cases} -\Delta h_n \cdot e^{-(g/\varepsilon)} & (x,y)\ \varepsilon\ T_1 \\ -\dfrac{1}{\varepsilon^{N+1}}(\varepsilon\Delta z+az_x+bz_y+cz) = O\left(\dfrac{e^{-(g/\varepsilon)}}{\varepsilon^{N+1}}\right) & \text{as } \varepsilon \to 0^+ \text{ in } T_2 \\ 0 & (x,y)\ \varepsilon\ (D+\partial D-T_1-T_2) \quad . \end{cases}$$

Provided $\varepsilon > 0$ is sufficiently small, the R.H.S. is $O(1)$ in $D+\partial D$. Hence we can write the problem for Z as follows:

PDE $\quad \varepsilon\Delta Z + aZ_x + bZ_y + cZ = O(1)$ uniformly in $D+\partial D$, $\varepsilon > 0$

BC $\quad Z = 0 \qquad (x,y)\ \varepsilon\ \partial D$.

By the maximum principle of theorem 3.2, we have that

$$Z(x,y;\varepsilon) = O(1) \quad \text{as} \quad \varepsilon \to 0^+$$

uniformly in $D+\partial D$.

9. A Refinement of the General Case of §6

In section 6, we have a solution of problem p_2 in a regular quadrilateral Q of the form:

$$u(x,y;\varepsilon) = U(x,y) + z(x,y;\varepsilon) + \varepsilon Z(x,y;\varepsilon) + w(x,y;\varepsilon) \qquad (9.1)$$

where (i) U is the solution of the reduced equation,

(ii) z is the boundary layer term and in the region T_1 is given by $z = e^{-(g/\varepsilon)}h$,

(iii) Z is the remainder term associated with this form of the boundary layer and is $O(1)$ uniformly in $D+\partial D$,

(iv) w is the remainder term introduced by the non-homogeneous term d in the equation (2.1) and the initial condition $U = \phi$ on Γ_o, and is $O(\varepsilon)$.

In this section we increase the order of our approximation from 1 to N+1 in ε as $\varepsilon \to 0^+$ in a regular quadrilateral Ω by assuming a solution of form

$$u(x,y;\varepsilon) = U_o(x,y) + \sum_{n=1}^{N} \varepsilon^n U_n(x,y) + z(x,y;\varepsilon) + \varepsilon^{N+1} Z(x,y;\varepsilon) + \varepsilon^N w \quad (9.2)$$

where (i) u is the solution to problem P_2 under the hypothesis H_1' rather than H_1,

(ii) z, Z, w play the same role as in §6 although they have different formulations here,

(iii) U_o is the same as U in §6,

(iv) $\sum_{n=1}^{N} \varepsilon^n U_n$ is a term whose introduction is necessary in order that the above terms have the properties stated. (More precisely the counter part of (6.3) above, which is (9.5) below, requires a term of order N+1 in ε on the R.H.S.)

<u>Theorem 9.1</u>. Given H_1', H_2 (or H_2 alternate) and problems P_1 and P_2 with solutions $U_o(x,y)$ and $u(x,y;\varepsilon)$, respectively. Then in any regular quadrilateral Ω with sides Γ_o and Γ_1 as given by (4.1) and (4.2) in $D+\partial D$ we have that w is given by (9.2) where

(i) $Z = O(1)$ uniformly in $D+\partial D$ and $Z = 0$ on ∂D.

(ii) $w = O(\varepsilon)$ uniformly in Q as $\varepsilon \to 0^+$ and $w = 0$ on Γ_o and Γ_1.

(iii) $z = e^{-\frac{g(x,y)}{\varepsilon}} \sum_{n=o}^{N} \varepsilon^n h_n(x,y)$ on Γ_1 and at points in Q near Γ_1 (9.3)

where (a) $g \in C^{5+2N}(Q)$, $h_n \in C^{3+2(N-n)}(Q)$,

(b) $g = 0$, $h_o = \psi - U$, $h_n = -U_n$ $(n = 1,2,\ldots,N)$ on Γ_1,

(c) $g > 0$ inside Q near Γ_1.

(iv) $U_n \in C^{2+2(N-n)}(Q)$.

$\delta_1 > 0$ such that $z = O(e^{-\delta_1/\varepsilon})$ as $\varepsilon \to 0^+$ uniformly in that part of Q where (9.3) no longer holds.

Proof. We shall use the notation of §6 again except where noted. The substitution of (9.2) into (2.2) yields

$$\varepsilon \sum_{n=o}^{N} \varepsilon^n \Delta U_n + \sum_{n=o}^{N} \varepsilon^n \left(a \frac{\partial U_n}{\partial x} + b \frac{\partial U_n}{\partial x} + c\, U_n\right)$$

$$+ \frac{1}{\varepsilon} e^{-(g/\varepsilon)}[(g_x^2 + g_y^2) - (ag_x + bg_y)] \sum_{n=o}^{N} \varepsilon^n h_n$$

$$+ e^{-(g/\varepsilon)} \sum_{n=o}^{N} \varepsilon^n \left[(a-2g_x) \frac{\partial h_n}{\partial x} + (b-2g_y) \frac{\partial h_n}{\partial y} + (c-\Delta g) h_n\right]$$

$$+ e^{-(g/\varepsilon)} \sum_{n=o}^{N} \varepsilon^{n+1} h_n + \varepsilon^{N+1}[\varepsilon \Delta Z + aZ_x + bZ_y + cZ]$$

$$+ \varepsilon^N [\varepsilon\Delta w + aw_x + bw_y + cw] = d \quad . \tag{9.4}$$

As before U_o solves the reduced problem P_1 (and is thus the same as U in §6). $U_n(x,y)$ solves problem P''_n $(n = 1,2,\ldots,N)$:

$$\text{PDE} \qquad a\frac{\partial U_n}{\partial x} + b\frac{\partial U_n}{\partial y} + cU_n = -\Delta U_{n-1} \qquad \text{in} \quad \overline{Q}+\partial\overline{Q}$$

$$\text{IC} \qquad U_n = 0 \qquad \text{on} \quad \overline{\Gamma}_o \quad .$$

The function $g(x,y)$ is again the same as in §5 and $h_o(x,y)$ agrees with $h(x,y)$ of §5 . And the functions $h_n(x,y)$ solve problem P'''_n $(n = 1,2,\ldots,N)$:

$$\text{PDE} \qquad (a-2g_x)\frac{\partial h_n}{\partial x} + (b-2g_y)\frac{\partial h_n}{\partial y} + (c-\Delta g)h_n = -\Delta h_{n-1} \quad \text{in} \quad \overline{Q} \quad \text{near} \quad \overline{\Gamma}_1$$

$$\text{IC} \qquad h_n = -U_n \qquad \text{on} \quad \overline{\Gamma}_1$$

for the function $Z(x,y;\varepsilon)$ we have the problem

$$\text{PDE} \qquad \varepsilon\Delta Z+aZ_x+bZ_y+cZ = \begin{cases} -\Delta h_N\, e^{-(g/\varepsilon)} & \text{in} \quad T_1 \\ -\dfrac{1}{\varepsilon^{N+1}}[\varepsilon\Delta z+az_x+bz_y+cz] & \text{in} \quad T_2 \\ 0 \quad \text{in} \quad (Q+\partial Q) - (T_1+T_2) & \end{cases}$$

$$\text{BC} \qquad Z = 0 \qquad \text{on} \quad \partial\overline{Q} \quad .$$

Note that while it is natural to set $Z = 0$ on $\overline{\Gamma}_o$ and $\overline{\Gamma}_1$ we arbitrarily set $Z = 0$ on the balance of $\partial\overline{Q}$. Further, by choosing ε as given in §5 sufficiently small, we can insure that the R.H.S. of the PDE is $O(1)$. So that by the maximum principle $Z = O(1)$ in $\overline{Q}$.

Lastly, as $u, \sum_{n=o}^{N} \varepsilon^n U_n$, z, Z are all defined, so is w as the solution of the problem

$$\text{PDE} \qquad \varepsilon\Delta w + aw_x + bw_y + cw = -\varepsilon\Delta U_N \quad \text{in } \overline{Q}+\partial\overline{Q} \tag{9.5}$$

$$\text{BC} \qquad w = 0 \quad \text{on } (\overline{\Gamma}_o + \overline{\Gamma}_1) \ . \tag{9.5)'}$$

w in general is non-zero on $\partial\overline{Q} - (\overline{\Gamma}_o + \overline{\Gamma}_1)$. Since this is the same as (6.3) and (6.3)' with U_N here replacing U , the argument for w in §7 may be repeated to show that $w = O(\varepsilon)$ in $(\overline{Q}+\partial\overline{Q})$.

10. An Extension of the Special Case of §5

We wish to extend our problem to the case where

$$a(x,y;\varepsilon) = \sum_{m=o}^{M} \varepsilon^m a_m(x,y)$$

and b, c, d are similarly defined, as are the boundary conditions for u on ∂D . The problem P_7 for $u(x,y;\varepsilon)$ is

$$\text{PDE} \qquad \varepsilon\Delta u + au_x + bu_y + cu = 0 \qquad (x,y) \in D \tag{10.1}$$

$$\text{BC} \qquad u = \begin{cases} \theta(s;\varepsilon) & (x,y) \in \Gamma_1 \\ 0 & (x,y) \in (\partial D - \Gamma_1) \end{cases}$$

where we assume that

$$\text{(i)} \quad \theta(s;\varepsilon) = \sum_{n=o}^{N} \varepsilon^n \theta_n(s) \text{ with } \theta_n \in C^{3+2(N-n)}(\Gamma_1)$$

(ii) $\theta \equiv 0$ over a small segment at each end of Γ_1 .

Note that the associated reduced problem has the solution $U(x,y;\varepsilon) \equiv 0$. We set

$$u(x,y;\varepsilon) = z(x,y;\varepsilon) + \varepsilon^{N+1} Z(x,y;\varepsilon) \qquad (10.2)$$

where $z = e^{-(g(x,y)/\varepsilon)} \sum_{n=o}^{N} \varepsilon^n h_n(x,y)$ with $N \geq M$

$$Z = \sum_{m=o}^{M} \varepsilon^m Z_m(x,y;\varepsilon) \quad .$$

Substitute (10.2) into (10.1) to get

$$e^{-(g/\varepsilon)} \sum_{n=o}^{N} \varepsilon^{n+1}[\Delta h_n - \frac{1}{\varepsilon} \Delta g \cdot h_n - \frac{2}{\varepsilon}(g_x h_{n,x} + g_y h_{n,y}) + \frac{1}{\varepsilon^2} (g_x^2 + g_y^2) h_n]$$

$$+ e^{-(g/\varepsilon)} \sum_{m=o}^{M+N} \sum_{n=o}^{m} \varepsilon^m (a_{m-n} h_{n,x} + b_{m-n} h_{n,y} + c_{m-n} h_n)$$

$$- e^{-(g/\varepsilon)} \sum_{m=o}^{M+N} \sum_{n=o}^{m} \varepsilon^{m-1} (a_{m-n} g_x + b_{m-n} g_y) h_n$$

$$+ \varepsilon^{N+2} \sum_{m=o}^{M} \varepsilon^m \Delta Z_m + \varepsilon^{N+1} \sum_{m=o}^{2M} \sum_{n=o}^{m} \varepsilon^m (a_{m-n} Z_{n,x} + b_{m-n} Z_{n,y} + c_{m-n} Z_n) = 0 \quad .$$

$g(x,y)$ solves

PDE $\qquad g_x^2 + g_y^2 - (a_o g_x + b_o g_y) = 0$

IC $\qquad g = 0 \qquad$ on Γ_1

$g > 0$ in Ω near Γ_1. Hence g is the same as in problem P_4. The coefficient of $\varepsilon^n e^{-(g/\varepsilon)}$, for $n = 0,1,\ldots,N$ yields the following equation for h_n:

$$(a_o-2g_x)h_{n,x} + (b_o-2g_x)h_{n,y} + (c_o-a_1g_x-b_1g_y-\Delta g)h_n$$

$$= -\Delta h_{n-1} - \sum_{k=1}^{n} (a_k h_{n-k,x} + b_k h_{n-k,y} + c_k h_{n-k})$$

$$+ \sum_{k=1}^{n} (a_{k+1}g_x + b_{k+1}g_y)h_{n-k} ,$$

with initial conditions $h_n = \theta_n$ on Γ_1. The coefficient of ε^{N+1} is

$$\varepsilon\Delta Z_o + a_o Z_{o,x} + b_o Z_{o,y} + c_o Z_o$$

$$= \left[\sum_{k=1}^{M-1} (a_{k+1}g_x+b_{k+1}g_y)h_{N+1-k} - \sum_{k=1}^{M} (a_k h_{N+1-k,x}+b_k h_{N+1-k,y} \right.$$

$$\left. +c_k h_{N+1-k}) - \Delta h_N \right] e^{-(g/\varepsilon)} \tag{10.3}$$

The coefficient of ε^{N+1+m}, for $m = 1,\ldots,M-1$, yields for Z_m:

$$\varepsilon\Delta Z_m + a_o Z_{m,x} + b_o Z_{m,y} + c_o Z_m = e^{-(g/\varepsilon)} \left[\sum_{k=1}^{M-1} (a_{k+1}g_x+b_{k+1}g_y)h_{N+1+m-k} \right.$$

$$\left. - \sum_{k=1}^{M} (a_k h_{N+1+m-k,x} + b_k h_{N+1+m-k,y} + C_k h_{N+1+m-k,y}) \right]$$

$$- \sum_{k=1}^{n} (a_k Z_{m-k,x} + b_k Z_{m-k,y} + c_k Z_{m-k}) \tag{10.4}$$

Lastly, the partial differential equation for Z_M is

$$\varepsilon\Delta Z_M + a_o Z_{M,x} + b_o Z_{M,y} + c_o Z_M = - \sum_{k=1}^{M} [a_k Z_{M-k,x} + b_k Z_{M-k,y} + c_k Z_{M-k}]$$

$$- \sum_{m=1}^{M} \varepsilon^m \sum_{k=m}^{M} [a_k Z_{M+m-k,x} + b_k Z_{M+m-k,y} + c_k Z_{M+m-k,y}] \ . \qquad (10.5)$$

Now in the region T_2 we define

$$z(\sigma,\tau;\varepsilon) = \sum_{n=o}^{N} \varepsilon^n z_n(\sigma,\tau;\varepsilon) \ ,$$

where

$$z_n(\sigma,\tau;\varepsilon) = (1 - \frac{\tau}{\tau_1})^3 \sum_{m=o}^{2} \tau^m z_{m\ n}(\sigma;\varepsilon)$$

continues the solution $e^{-(g/\varepsilon)} h_n$ from T_1 into T_2 such that $z \in C^2(T_1 + T_2)$. Thus the remainder function Z_o satisfies the PDE (10.3) in the region T_1 , (10.6) below in the region T_2 , and (10.7) in the region $D - (T_1 + T_2)$ where we have

$$\varepsilon\Delta Z_o + a_o Z_{o,x} + b_o Z_{o,y} = - \frac{1}{\varepsilon^{N+1}} [\varepsilon\Delta z + az_x + bz_y + cz]$$

$$= 0 \left(\frac{e^{-(\delta_1/\varepsilon)}}{\varepsilon^{N+1}} \right) \quad \text{as} \quad \varepsilon \to 0^+ \qquad (10.6)$$

and

$$\varepsilon\Delta Z_o + a_o Z_{o,x} + b_o Z_{o,y} + c_o Z_o = 0 \qquad (10.7)$$

with the boundary condition $Z_o = 0$ on ∂D. Hence we have that $Z_o = O(1)$ as $\varepsilon \to 0^+$ uniformly in the region D.

In similar fashion, we have that Z_m $(m = 1,2,\ldots,M-1)$ satisifes (10.4) in the region T_1 while in the region $D - T_1$ we have

$$\varepsilon\Delta Z_m + a_o Z_{m,x} + b_o Z_{m,y} + c_o Z_m = - \sum_{k=1}^{m} (a_k Z_{m-k,x} + b_k Z_{m-k,y} + c_k Z_{m-k}) .$$

Since the boundary conditions are $Z_m = 0$ on ∂D, we have, by induction, that $Z_m = O(1)$ as $\varepsilon \to 0^+$ uniformly in the region D.

Finally, Z_M satisfies (10.5) in the region D so that, with the boundary condition $Z_M = 0$ on ∂D, we have the $Z_M = O(1)$ as $\varepsilon \to 0^+$ uniformly in the region D. Thus we conclude that the remainder term Z is $O(1)$ uniformly in D as $\varepsilon \to 0^+$. As before, we assume $a, b, c, x(s), y(s) \varepsilon C^{6+2N}$ so that $g \varepsilon C^{5+2N}$, $h_n \varepsilon C^{3+2(N-n)}$ for $n = 0,1,\ldots,N$, and $Z \varepsilon C^2$ where $\theta_n \varepsilon C^{3+2(N-n)}$.

11. An Extension of the General Case of §6

As in §10, we wish to have the coefficients a, b, c, d expressed as polynomials of order M in terms of the parameter ε. We shall then construct a solution of form

$$u(x,y;\varepsilon) = U_o(x,y) + \sum_{n=1}^{N} \varepsilon^n U_n(x,y) + z(x,y;\varepsilon) + \varepsilon^{N+1} Z(x,y;\varepsilon) + \varepsilon^N w(x,y;\varepsilon) .$$

The discussion is almost the same as in §9 except now U_n solves $(n = 1,2,\ldots,N)$

$$\text{PDE} \qquad a_o U_{n,x} + b_o U_{n,y} + c_o U_n = -\Delta U_{n-1} - \sum_{k=1}^{n} (a_k U_{n-k,x} + b_k U_{n-k,y} + c_k U_{n-k}) \quad \text{in} \quad (Q+\partial Q)$$

$$\text{IC} \qquad U_n = 0 \qquad \text{on} \quad \Gamma_o .$$

For h_n, the initial condition becomes $h_n = \theta_n - U_n$ on Γ_1. Throughout the entire region D we add to the R.H.S. of equations (10.3) and (10.4) that is the coefficient of ε^{N+m} $(m = 1,2,\ldots,M)$ the term[1]

$$\sum_{k=m}^{M} (a_k U_{N+m-k,x} + b_k U_{N+m-k,y} + c_k U_{N+m-k}) .$$

Lastly, as the problem solved by w in §9 is unchanged, we have once again the $w = 0(\varepsilon)$ uniformly in D as $\varepsilon \to 0^+$.

---

(1) Note that this does not change the order of Z_m $(m = 0,1,\ldots,M-1)$.

CHAPTER III

LINEAR PARABOLIC DIFFERENTIAL EQUATION CONTAINING A SMALL PARAMETER

1. Introduction

Aronson in [1] investigates the first boundary value problem for the parabolic equation

$$\varepsilon u_{xx} + a(x,y)u_x - b(x,y)u_y + c(x,y)u = d(x,y) \tag{1.1}$$

in D, an open region in xy-plane bounded above by the line $y = y_2$ and below by the concave-upward curve Γ, where $\varepsilon > 0$ is a parameter and $b(x,y) > 0$. If the given data is sufficiently smooth, there exists a unique solution $u(x,y;\varepsilon)$ satisfying equation (1.1) and the boundary condition $u = \phi$ on Γ.(1) It is shown that the solution $u(x,y;\varepsilon)$ has the form $u(x,y;\varepsilon) = U(x,y) + z(x,y;\varepsilon) + w(x,y;\varepsilon)$ in a regular multilateral M (which will be defined later) where U is the solution of

$$a(x,y)U_x - b(x,y)U_y + c(x,y)U = d(x,y) \quad (x,y) \in D \tag{1.2}$$

$$u = \phi \quad (x,y) \in \Gamma .$$

w is $O(\varepsilon)$ in M and z is a boundary layer term similar to that obtained by Levinson [7].

---

(1) See [4], p. 41.

In §§7, 8, 9 and 10, we shall extend Aronson's results as we have extended, in the last chapter, Levinson's results.

2. Statement of the Problem

Let D_0 be an open region in the xy-plane, and let $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ be two closed arcs in D_0 which are defined for $y_1 \leq y \leq y_2$ and do not meet for $y > y_1$. Denote by $\overline{D} \subset D_0$[(1)] the closed region bounded above by a segment $\overline{\Gamma}_4$ of $y = y_2$, below by a segment $\overline{\Gamma}_1$ of $y = y_1$, and laterally by $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ (See Fig.3·1). Let s denote arc length on $\partial D = \Gamma_1 + \Gamma_4 + \overline{\Gamma}_2 + \overline{\Gamma}_3$ measured positively in the counter-clockwise direction. We consider the problem

$$P_1: \qquad \text{PDE} \qquad \varepsilon u_{xx} + au_x - bu_y + cu = d \qquad (x,y) \in (D + \partial D)$$

$$\text{IC} \qquad u = \begin{cases} \phi_1 & \text{on } \Gamma_1 \\ \phi_2 & \text{on } \overline{\Gamma}_2 \\ \phi_3 & \text{on } \overline{\Gamma}_3 \end{cases}$$

where ϕ_1, ϕ_2, $\phi_3 \in C^4$ in s and $\phi_2 = \phi_1$, $\phi_3 = \phi_1$ at $\overline{\Gamma}_2 \cap \overline{\Gamma}_1$, $\overline{\Gamma}_3 \cap \overline{\Gamma}_1$, respectively.

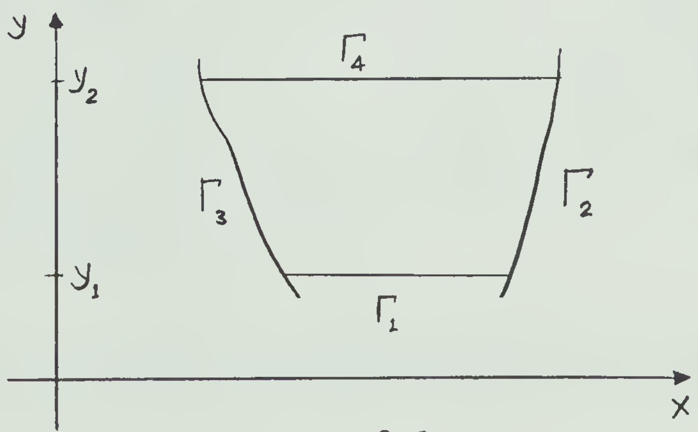

Figure 3.1

(1) If A is an open set, then by $\overline{A}$ we mean the closure of A.

Let us also assume the following and call them hypothesis H_1:

(A) $a, b, c, d \in C^4(D_0)$

(B) $\varepsilon > 0$, $b(x,y) > m > 0$ in D_0

(C) $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ are of class C^4 with respect to s and are nowhere horizontal.

The characteristics of the reduced partial differential equation (1.2) are given by

$$\frac{dx}{dt} = -a(x,y) \; , \qquad \frac{dy}{dt} = b(x,y) \; . \tag{2.1}$$

In view of H_1 parts (A) and (B), there exists a unique solution of (2.1) through every point of D_0 which can be continued as far as the boundary of D_0 and which is continuous with respect to its initial values.

Let I be a closed sub-arc of Γ_i $(i = 1,2,3)$ such that no characteristic is tangent to I and such that every characteristic which starts on I enters D for increasing t, and first leaves D via a closed subarc $J \subset \Gamma_2 + \Gamma_3 + \overline{\Gamma}_4 - I$. Let no characteristic starting on I be tangent to J. We call the closed region bounded by I, J and the characteristics joining their end-points a regular multi-lateral M (see Fig. 3.2).

If ∂D is given parametrically as $x = x(s)$, $y = y(s)$, then

$$\Delta = -ay_s - bx_s \tag{2.2}$$

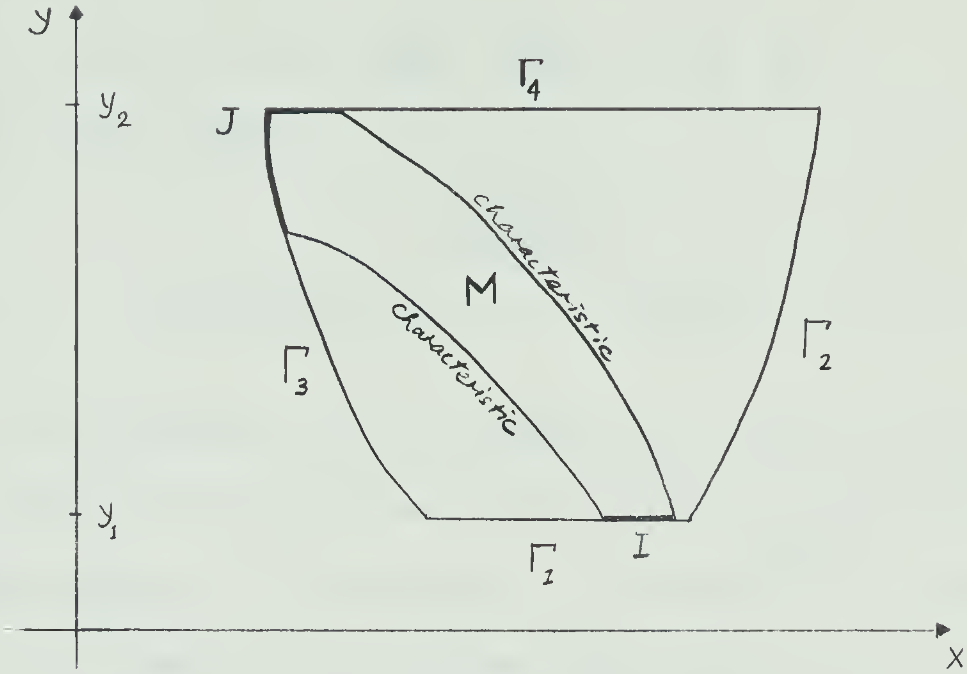

Figure 3.2

is (i) equal to 0 at any point Γ is tangent to a characteristic of (1.2), (ii) is less than zero on I, and (iii) is greater than zero on J. Thus every point in $D + \Gamma$ which does not lie on a characteristic which is tangent to Γ or on the characteristics through $\overline{\Gamma}_1 \cap \overline{\Gamma}_i$ $(i = 2,3)$ can be included in a regular multilateral.

Main Theorem. Let H_1 hold. Consider the problem

$$P_2 \qquad \text{PDE} \qquad aU_x - bU_y + cU = d \qquad (x,y) \in M \tag{2.3}$$

$$\text{IC} \qquad U = u \qquad (x,y) \in I$$

and problem P_1 above. Then in the multilateral M,

$$u(x,y;\varepsilon) = U(x,y) + z_2(x,y;\varepsilon) + z_3(x,y;\varepsilon) + w(x,y;\varepsilon) \tag{2.4}$$

where (i) $w = 0(\varepsilon)$ uniformly in M as $\varepsilon \to 0^+$, $w = 0$ on

$M \cap \{\Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3\}$,

(ii) if $J \cap \overline{\Gamma}_i = 0$ $(i = 2,3)$, then $z \equiv 0$; if $J \cap \overline{\Gamma}_i \neq 0$ $(i = 2,3)$, then

$$z_i(x,y;\varepsilon) = e^{-\frac{g_i(x,y)}{\varepsilon}} \cdot h_i(x,y) \qquad (2.5)$$

at point of M on or near $J \cap \overline{\Gamma}_i$, where $g_i = 0$ on $J \cap \overline{\Gamma}_i$, $g_i > 0$ off $J \cap \overline{\Gamma}_i$, $h_i = u - U$ on $J \cap \overline{\Gamma}_i$ and $g_i, h_i \in C^2$. Moreover, there exists $\delta > 0$ such that $z_i = 0\,(e^{-(\delta/\varepsilon)})$ uniformly in that part of M where (2.5) no longer holds, and $z_i = 0$ on I.

3. Maximum Principles

Let D be the region given at the beginning of §2, and consider

$$L_\varepsilon[u] \equiv \varepsilon u_{xx} + \alpha(x,y)u_x + \beta(x,y)u_y + \gamma(x,y)u = \delta(x,y) \qquad (3.1)$$

in D. $u(x,y;\varepsilon)$ is said to be a regular solution of $L_\varepsilon(u) = \delta$ in $D + \partial D$ if u satisfies the differential equation in D and is continuous in $D + \partial D$ and if $u_x, u_y, u_{xx} \in C(D + \Gamma_4)$.

Lemma 3.1. Assume

(I) $\alpha, \beta, \gamma, \delta \in C^1(D_0)$.

(II) $\varepsilon > 0$ and $\beta > m > 0$ in D_0.

(III) $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ are piecewise smooth and nowhere horizontal.

(IV) Functions $\phi_i \in C^1$ are given on $\overline{\Gamma}_i$ $(i = 1,2,3)$ such that $\phi_i = \phi_1$ at $\overline{\Gamma}_i \cap \overline{\Gamma}_1$ $(i = 2,3)$.

Then, for fixed ε, (3.1) has a regular solution in $D + \partial D$, which is equal to ϕ_i on $\overline{\Gamma}_i$ $(i = 1,2,3)$.

Lemma 3.2. Let u be a regular solution of $L_\varepsilon[u] = 0$ in $D + \partial D$, where $\varepsilon > 0$, $\beta > 0$, $\gamma < 0$ in $D + \partial D$, and α, β, γ are bounded in $D + \partial D$. Then u cannot take on a positive maximum or a negative minimum in $D + \Gamma_4$.

The proofs of Lemmas 3.1 and 3.2 are given in [4], Chapter 2.

Lemma 3.3. Let $u(x,y;\varepsilon)$ be a regular solution of (3.1), where $\varepsilon > 0$, $\beta > 0$ in $D + \partial D$, and α, β, γ are bounded in $D + \partial D$. If $|\delta(x,y)| \leq n$ in $D + \partial D$ and $|u| \leq n$ on $\Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3$, then there exists a constant $K > 0$, independent of ε, such that $|u| \leq K\, n$ in $D + \partial D$.

Proof. Let $u = e^{ky}\, v(x,y;\varepsilon)$ where $k > 0$ is a constant such that $\gamma(x,y) - k \cdot \beta(x,y) \leq - \mu < 0$ in $D + \partial D$ for some constant $\mu > 0$. Then (3.1) becomes

$$\varepsilon v_{xx} + \alpha v_x - \beta v_y + (\gamma - k\beta)v = \delta \cdot e^{-ky} \tag{3.2}$$

Now v may not have a positive maximum nor negative minimum in $D + \Gamma_4$ such that $|v| > \frac{n}{\mu} e^{-ky_1}$. For suppose it did have a positive maximum at $P_o \in D$ such that $v > \frac{n}{\mu} e^{-ky_1}$. Then

$$v_{xx}(P_o) \leq 0, \quad v_x(P_o) = v_y(P_o) = 0.$$

And

$$[\varepsilon v_{xx} + (\gamma - k\beta)v]\Big|_{P_o} < -\mu \cdot \frac{n}{\mu} \cdot e^{-ky_1} = -ne^{-ky_1},$$

i.e. the L.H.S. of (3.2) is less than $-ne^{-ky_1}$. But the R.H.S. of (3.2) is greater than or equal to $-ne^{-ky_1}$.

Similarly for a negative minimum, $v < -\frac{n}{\mu} e^{-ky_1}$.

Now $|u| \leq n$ on $\Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3$ implies $|v| \leq ne^{-ky_1}$ on $\Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3$. Since $|v| \leq \frac{n}{\mu} e^{-ky_1}$ in $D + \Gamma_4$, then $|v| \leq n \cdot e^{-ky_1} \cdot \max\{1, \frac{1}{\mu}\}$ in $D + \Gamma_1 + \Gamma_4 + \overline{\Gamma}_2 + \overline{\Gamma}_3$. Therefore $|u| \leq n \cdot e^{k(y-y_1)} \cdot \max\{1, \frac{1}{\mu}\} \leq k \cdot n$ in $(D + \partial D)$.

4. The Statement and Proof of the Main Theorem in Special Case

Let $J^* \subset \Gamma_i$ $(i = 2,3)$ be any closed sub-arc such that

$$\Delta > 0 \quad \text{on} \quad J^* \tag{4.1}$$

where Δ is given by (2.2). Let $\theta(s) \in C^2(J^*)$ such that $\theta(s) \equiv 0$ in a neighborhood of each end-point of J^*. Consider the problem

$$P_3: \qquad \text{PDE} \qquad \varepsilon u_{xx} + au_x - bu_y + cu = 0 \quad (x,y) \in M \tag{4.2}$$

$$\text{IC} \qquad u = \begin{cases} \theta(s) & \text{on } J^* \\ 0 & \text{on } \Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3 - J^*. \end{cases}$$

The associated reduced equation of (4.2) with homogeneous initial condition has solution $U(x,y) \equiv 0$ in $D + \partial D$.

<u>Theorem 4.1</u>. Given H_1 and problem P_3. Then in any regular

multilateral M there exists a regular solution of form

$$u(x,y;\varepsilon) = z(x,y;\varepsilon) + \varepsilon\, Z(x,y;\varepsilon) \tag{4.3}$$

where

$$\text{(i)}\quad Z(x,y;\varepsilon) = \begin{cases} O(1) & \text{uniformly in } M \text{ as } \varepsilon \to 0^+ \\ 0 & \text{on } \Gamma_1 + \bar{\Gamma}_2 + \bar{\Gamma}_3 \end{cases}$$

$$\text{(ii)}\quad z = e^{-\frac{g(x,y)}{\varepsilon}} \cdot h(x,y) \tag{4.4}$$

on J^* and at points in M near J^* where $g,\ h \in C^2(M)$; $g = 0$, $h = \theta$ on J^* and $g > 0$ inside M near J^*. There exists $\delta > 0$ such that $z = 0\ (e^{-(\delta/\varepsilon)})$ as $\varepsilon \to 0^+$ uniformly in that part of M where (4.4) no longer holds.

<u>Proof</u>. The proof paraphases the proof in §5, Chapter II except for the appropriate minor changes noted below.

By lemma 3.3, $u(x,y;\varepsilon)$ is uniformly bounded in $M + \partial M$ as $\varepsilon \to 0^+$. If we substitute (4.3) into (4.2) we obtain

$$\frac{h}{\varepsilon}\,[g_x^{\,2} - ag_x + bg_y]\, e^{-(g/\varepsilon)} + [(a - 2g_x)h_x - bhy + (c - g_{xx})h]\, e^{-(g/\varepsilon)}$$
$$+ \varepsilon h_{xx}\, e^{-(g/\varepsilon)} + \varepsilon[\varepsilon\, Z_{xx} + aZ_x - bZ_y + cZ] = 0.$$

We obtain $g(x,y)$ by setting the coefficient of $\frac{h}{\varepsilon}\, e^{-(g/\varepsilon)}$ equal to zero to get

P_4: PDE $g_x^2 - ag_x + bg_y = 0$ in M near J^*

IC $g = 0$ on J^* and $g > 0$ in M near J^*.

We obtain $h(x,y)$ by setting the coefficient of $e^{-(g/\varepsilon)}$ equal to zero to get

$$P_5: \quad \text{PDE} \quad (a - 2g_x)h_x - bh_y + (c - g_{xx})h = 0 \quad \text{in } M \text{ near } J^*$$

$$\text{IC} \quad h = \theta \quad \text{on } J^*.$$

The method of solution for $g(x,y)$ of P_4 and $h(x,y)$ of P_5 is the same manner as that in §5, Chapter II. But instead of (5.4), Chapter II, we have that the system of characteristic equations for P_4 is

$$\frac{dx}{dt} = 2p - a, \quad \frac{dy}{dt} = b,$$

$$\frac{dg}{dt} = p^2, \quad \frac{dp}{dt} = a_x p - b_x q,$$

$$\frac{dq}{dt} = a_y p - b_y q, \quad p^2 = ap - bq;$$

and we obtain

$$p = (ay_s + bx_s)/y_s,$$

$$q = -x_s(ay_s + bx_s)/y_s^2.$$

Instead of (5.6), Chapter II, we have here that the characteristic system for P_5 is

$$\frac{dx}{dt} = -a + 2q_x, \quad \frac{dy}{dt} = b,$$

$$\frac{dh}{dt} = (c - g_{xx})h.$$

The equations that Z satisfies in various regions in $D + \partial D$ are the same except for the change of the operator from elliptic to parabolic.

Also throughout the whole proof we replace Q and Γ_1 by M and J^*, respectively.

5. The Proof of Main Theorem in the General Case

Let M be a fixed regular multilateral and let J be given in terms of arc length for $s_1 \leq s \leq s_2$. Since each characteristic of (1.2) in M meets J exactly once, the coordinate s denotes a unique characteristic in M. Since (2.2) involves strict inequalities, and since no characteristic in M meets $\partial D - (I + J)$, we can always extend I and J to obtain an enlarged regular multilateral M_3 which contains the characteristics $s_1 - 3\delta \leq s \leq s_2 + 3\delta$ for some $\delta > 0$. Thus $M = M_0$ can always be embedded in a series of regular multilaterals M_k $(k = 0,1,2,3)$ which contain the characteristics $s_1 - k\delta \leq s \leq s_2 + k\delta$. Let I_k and J_k denote the non-characteristic boundaries of M_k. Let $U(x,y)$ be the solution of P_2 which takes on the values of $u(x,y;\varepsilon)$ on I_2. Then U exists uniquely in M_2 and is of class $C^4(M_2)$.

Now we define w in (2.4) and distinguish the following three cases:

Case 1. $J \subset \Gamma_4$. Then we define $w(x,y;\varepsilon)$ by $u(x,y;\varepsilon) = U(x,y) + w(x,y;\varepsilon)$ where u is the unique regular solution of problem P_1 in $D + \partial D$.

Case 2. $J \subset \Gamma_i$ $(i = 2,3)$. Let $\theta(s) = u - U$ on J_2 and let θ be continuous over $J_3 - J_2$ such that $\theta \in C^2(J_3)$ and $\theta \equiv 0$ in a neighborhood of the end-points of J_3. Let $\tilde{u}(x,y;\varepsilon)$ be

the regular solution as obtained in §4 to the problem

$$\text{PDE} \qquad \varepsilon u_{xx} + au_x - bu_y + cu = 0 \quad \text{in} \quad D + \partial D$$

$$\text{IC} \qquad u = \begin{cases} \theta & \text{on} \quad J_3 \\ 0 & \text{on} \quad \Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3 - J_3 \,. \end{cases}$$

We define w in M_2 by

$$u(x,y;\varepsilon) = U(x,y) + \tilde{u}(x,y;\varepsilon) + w(x,y;\varepsilon).$$

Case 3. $J \cap \overline{\Gamma}_4 \neq 0$, $J \not\subset \Gamma_4$. Let $\overline{D}^* \subset D_o$ be the closed region bounded below by Γ_1, above by a segment of the line $y = y_3$ where $y_3 > y_2$, and laterally by the arcs $\overline{\Gamma}_i^*$ $(i = 2,3)$ which are obtained by continuing the $\overline{\Gamma}_i$ above $y = y_2$ in such a way that hypothesis $H_1(C)$ continues to hold. Now $J_k \cap \overline{\Gamma}_i \neq 0$ $(k = 0,1,2,3)$ for $i = 2$ or $i = 3$ or $i = 2$ and 3. For such an i we have $J_2 \cap \overline{\Gamma}_i \subset \Gamma_i^*$ and $\Delta > 0$ on $J_2 \cap \overline{\Gamma}_i$. By continuity, there exists a closed subarc $J_i^* \subset \Gamma_i^*$ such that J_i^* coincides with $J_3 \cap \overline{\Gamma}_i$ below $\overline{\Gamma}_4 \cap \overline{\Gamma}_i$, which extends a finite distance above $\overline{\Gamma}_4 \cap \overline{\Gamma}_i$ and $\Delta > 0$ on J_i^*. Let $\theta_i(s) = u - U$ on $J_2 \cap \overline{\Gamma}_i$, and let θ_i be continued over $J_i^* - \overline{J}_2$ such that $\theta_i \in C^2(J_i^*)$ and $\theta_i \equiv 0$ in a neighborhood of each end-point of J_2^*. Let $\tilde{u}_i = \tilde{u}_i(x,y;\varepsilon)$ be the regular solution of the problem

$$\text{PDE} \qquad \varepsilon u_{xx} + au_x - bu_y + cu = 0 \quad (x,y) \in (D^* + \partial D^*)$$

$$\text{IC} \qquad u = \begin{cases} \theta_i & (x,y) \in J_i^* \\ 0 & (x,y) \in (\Gamma_1 + \overline{\Gamma}_2^* + \overline{\Gamma}_3^* - J_i^*). \end{cases}$$

Note that, as $(D^* + \partial D^*) \subset D_0$ and $\Delta > 0$ on $J_i^* \subset \Gamma_i^*$, the results of §4 can be applied to $\tilde{u}_i$. If $J_k \cap \overline{\Gamma}_i = 0$, we set $\tilde{u}_i \equiv 0$. We now define w in M_2 by

$$u(x,y;\varepsilon) = U(x,y) + \tilde{u}_2(x,y;\varepsilon) + \tilde{u}_3(x,y;\varepsilon) + w(x,y;\varepsilon).$$

These cases are shown in the Figure 3.3 below. The arc abc is a characteristic of (1.2) which is tangent to Γ_3 at b. Note that in

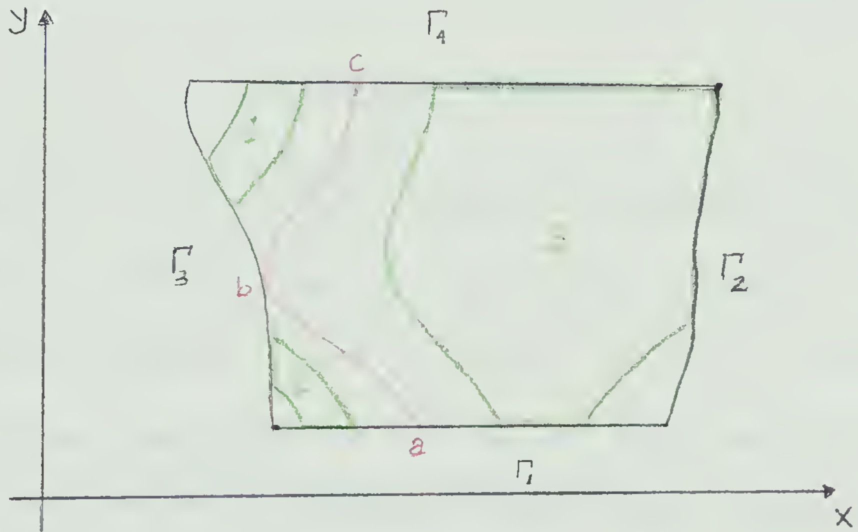

Figure 3.3

each case M can be embedded in a family of regular multilaterals of the same type.

In all the above cases, w solves the problem

$$\text{PDE} \qquad \varepsilon w_{xx} + a w_x - b w_y + c w = -\varepsilon U_{xx} \qquad (x,y) \in M_2$$

$$\text{IC} \qquad w = 0 \qquad (x,y) \in M_2 \cap [\Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3].$$

w is unspecified but bounded on the boundaries of M_2 that are characteristics. w is uniformly bounded in M_2 as $\varepsilon \to 0$, since u, $\tilde{u}_2$, $\tilde{u}_3$ are all uniformly bounded in $D + \partial D$ as $\varepsilon \to 0$ and U is also bounded.

In the next section we shall show that $w = 0(\varepsilon)$ uniformly in $M_o = M$ as $\varepsilon \to 0^+$. This, together with Theorem 4.1 in §4 will complete the proof of the main theorem in §2.

6. The Proof that $w = 0(\varepsilon)$

The proof in this section is analogous to §7, Chapter II. The partial differential equation

$$a(x,y)\,\sigma_x - b(x,y)\,\sigma_y = 0 \tag{6.1}$$

has the solution $\sigma(x,y) \in C^4(M_2)$ and $\sigma =$ constant are the characteristics of (1.2). Let $\sigma = \sigma_k^{(1)}$ and $\sigma = \sigma_k^{(2)}$ be the characteristics bounding M_k $(k = 0,1,2)$. Let y^* be the largest constant such that the line $y = y^*$ has at least one point in common with M_{k-1}. Let $\hat{M}_k$ be the closed region obtained by excluding the part of M_k above the line $y = y^*$. If $J_k \cap \bar{\Gamma}_4 \neq 0$, then $\hat{M}_k \equiv M_k$. If $J_k \cap \bar{\Gamma}_4 = 0$, then $M_{k-1} \subset \hat{M}_k \subset M_k$. These two cases are shown in Figure 3.4 and Figure 3.5 respectively.

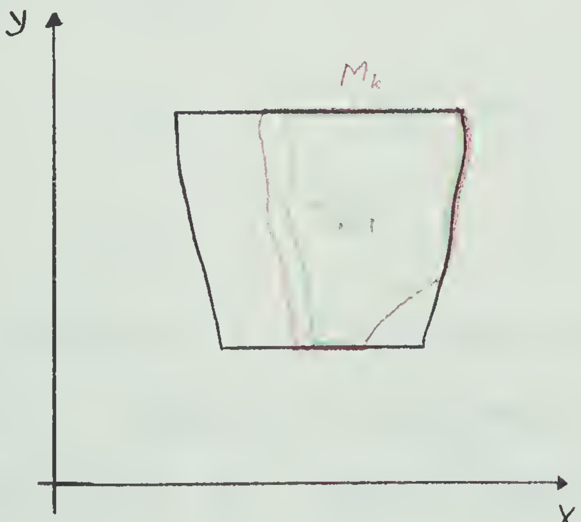

Figure 3.4

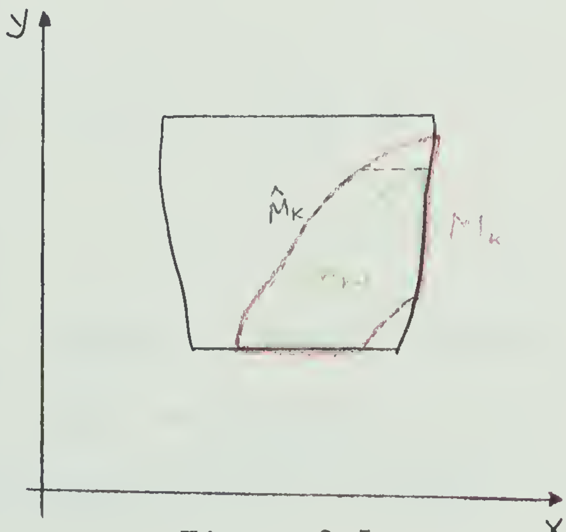

Figure 3.5

<u>Lemma 6.1</u>. Let H_1 hold. If $W(x,y;\varepsilon)$ is a regular solution of

$$\text{PDE} \qquad \varepsilon W_{xx} + aW_x - bW_y + cW = 0 \quad \text{in } \hat{M}_k$$

$$\text{IC} \quad \begin{cases} W = 0 & \text{on } \hat{M}_k \cap [\Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3] \\ |W| < K\varepsilon^n & \text{on } \sigma = \sigma_k^{(1)} \text{ and } \sigma = \sigma_k^{(2)} \text{ in } \hat{M}_k \end{cases}$$

where K is independent of ε and $c(x,y) < 0$ in $\hat{M}_k$, then

$$W = O(\varepsilon^{n+\frac{1}{2}}) \quad \text{uniformly in } M_{k-1} \text{ as } \varepsilon \to 0^+.$$

<u>Proof</u>. Let $\partial\hat{M}_k$ denote the boundary of $\hat{M}_k$ below the line $y = y^*$. Suppose $V = V(x,y;\varepsilon)$ is a regular solution of the problem

$$\text{PDE} \qquad \varepsilon V_{xx} + aV_x - bV_y + cV = 0 \quad \text{in } \hat{M}_k \tag{6.2}$$

$$\text{IC} \quad \begin{cases} V \geq 0 & \text{on } \partial\hat{M}_k \cap [\Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3] \\ V \geq K\varepsilon^n & \text{on } \sigma = \sigma_k^{(1)} \text{ and } \sigma = \sigma_k^{(2)} \text{ in } \hat{M}_k \end{cases}$$

with

$$V = O(\varepsilon^{n+\frac{1}{2}}) \quad \text{uniformly in } M_{k-1} \text{ as } \varepsilon \to 0^+.$$

By lemma 3.2, $V \geq 0$ in $\hat{M}_k$. Now consider the function $P(x,y;\varepsilon) = V - W$. Then P solves

$$\text{PDE} \qquad \varepsilon P_{xx} + aP_x - bP_y + cP = 0 \quad \text{in } \hat{M}_k$$

$$\text{IC} \qquad P \geq 0 \quad \text{on } \partial\hat{M}_k.$$

By lemma 3.2, $P \geq 0$ uniformly in $\hat{M}_k$. Therefore $V \geq W$ uniformly in $\hat{M}_k$. Similarly by considering $P = V + W$, we find that $W \geq -V$ uniformly in $\hat{M}_k$. Thus the lemma will be proved once a suitable V is

exhibited.

Consider the function

$$V(x,y;\varepsilon) = \sum_{\alpha=1}^{2} e^{-\gamma_k^{(\alpha)}} h^{(\alpha)}(x,y;\varepsilon) + \sqrt{\varepsilon}\, Y(x,y;\varepsilon) ,$$

where $\gamma_k^{(\alpha)} = \frac{1}{\sqrt{\varepsilon}} (\sigma - \sigma_k^{(\alpha)})^2$. Substitute V into (6.2) to get

$$\sum_{\alpha=1}^{2} e^{-\gamma_k^{(\alpha)}} [ah_x^{(\alpha)} - bh_y^{(\alpha)} + \{c + 4(\sigma - \sigma_k^{(\alpha)})^2\, \sigma_x^2\}\, h^{(\alpha)}]$$

$$+ \sqrt{\varepsilon}\, G(x,y;\varepsilon) + \sqrt{\varepsilon}\, (\varepsilon Y_{xx} + aY_x - bY_y + cY) = 0 ,$$

where $G(x,y;\varepsilon) = \sum_{\alpha=1}^{2} e^{-\gamma_k^{(\alpha)}} [\sqrt{\varepsilon}\, h_{xx}^{(\alpha)} - 4(\sigma - \sigma_k^{(\alpha)})\, \sigma_x h_x^{(\alpha)} -$

$$- 2\{\sigma_x^2 + (\sigma - \sigma_k^{(\alpha)})\sigma_{xx}\}\, h^{(\alpha)}] .$$

We take $h^{(\alpha)}$ to be the solution of

PDE $\quad ah_x^{(\alpha)} - bh_y^{(\alpha)} + [c + 4(\sigma - \sigma_k^{(\alpha)})^2\, \sigma_x^2]\, h^{(\alpha)} = 0$ in $\hat{M}_k$

IC $\quad h^{(\alpha)} = \frac{K\varepsilon^n}{N}$ on I_k

where $N > 0$ is a constant to be specified later. As in §7, Chapter II, $0 < N \leq 1$, $h^{(\alpha)} \geq K\varepsilon^n$ on $\sigma = \sigma_k^{(\alpha)}$ in $\hat{M}_k$, if we let $N = \min_{(\sigma,t)\,\varepsilon\, \hat{M}_k} e^{\int_0^t c dt}$. Also $Y = O(\varepsilon^n)$ uniformly in $\hat{M}_k$. Thus we have found the required V.

<u>Theorem 6.1</u>. Let H_1 hold. If $w(x,y;\varepsilon)$ is a regular solution of

$$\text{PDE} \qquad \varepsilon w_{xx} + aw_x - bw_y + cw = -\varepsilon U_{xx} \qquad (x,y) \in M_2$$

$$\text{IC} \qquad \begin{cases} w = 0 & \text{on} \quad M_2 \cap [\Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3] \\ |w| < B & \text{on} \quad \sigma = \sigma_2^{(1)} \quad \text{and} \quad \sigma = \sigma_2^{(2)} \quad \text{in} \quad M_2 \end{cases}$$

where B is independent of ε, then $w = 0(\varepsilon)$ uniformly in M_o as $\varepsilon \to 0^+$.

Proof. By lemma 3.1 there exists a regular solution $W_o(x,y;\varepsilon)$ of

$$\text{PDE} \qquad \varepsilon W_{o,xx} + aW_{o,x} - bW_{o,y} + cW_o = -\varepsilon U_{xx} \quad \text{in} \quad \hat{M}_2$$

$$\text{IC} \qquad W_o = 0 \quad \text{on} \quad \partial\hat{M}_2 .$$

By lemma 3.3, $W_o = 0(\varepsilon)$ uniformly in $\hat{M}_2$. Let $w = W_o + W$. Then W solves

$$\text{PDE} \qquad \varepsilon W_{xx} + aW_x - bW_y + cW = 0 \quad \text{in} \quad \hat{M}_2$$

$$\text{IC} \qquad \begin{cases} W = 0 & \text{on} \quad M_2 \cap [\Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3] \\ |W| < B & \text{on} \quad \sigma = \sigma_2^{(1)} \quad \text{and} \quad \sigma = \sigma_2^{(2)} \quad \text{in} \quad \hat{M}_2 . \end{cases}$$

By lemma 6.1 with $n = 0$ and $k = 2$ we have that $W = 0(\varepsilon^{1/2})$ uniformly in M_1; i.e. there exists $K > 0$, independent of ε, such that $|W| < K\sqrt{\varepsilon}$ in M_1. Since $\hat{M}_1 \subseteq M_1$, by lemma 6.1 with $n = \frac{1}{2}$ and $k = 1$, it follows that $W = 0(\varepsilon)$ uniformly in M_o as $\varepsilon \to 0^+$.

7. A Refinement of the Special Case of §4

Just as in the case of the elliptic problem, we wish to improve the order of the remainder term εZ from 1 to $N + 1$ in ε

as $\varepsilon \to 0^+$. To do so we must refine our hypothesis H_1 to H_1' :

(i) $a, b, c, d \in C^{4+2N}(D)$.

(ii) The functions, yielding x and y in terms of arc length specifying each boundary curve are now of class C^{4+2N} in arc length.

(iii) $\theta(s) \in C^{2N}(J^*)$ and the initial condition for u are also of class $C^{2N}(\Gamma)$.

Theorem 7.1. Given H_1' and problem P_3. Then in any regular multilateral M, there exist a regular solution of form

$$u(x,y;\varepsilon) = z(x,y;\varepsilon) + \varepsilon^{N+1} Z(x,y;\varepsilon) \tag{7.1}$$

where

$$Z(x,y;\varepsilon) = \begin{cases} 0(1) & \text{uniformly in } M \text{ as } \varepsilon \to 0^+ \\ 0 & \text{on } \Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3 \end{cases}$$

and

$$z = e^{-g(x,y)/\varepsilon} \sum_{n=o}^{N} \varepsilon^n h_n(x,y) \tag{7.2}$$

on J^* and at points of M near J^* where $g \in C^{3+2N}(M)$, $h_n \in C^{1+2(N-n)}(M)$; $g = 0$, $h_o = \theta$, $h_n = 0$ $(n = 1,2,\ldots,N)$ on J^*; $g > 0$ inside M near J^*. Moreover, there exists $\delta > 0$ such that $z = 0(e^{-\delta/\varepsilon})$ as $\varepsilon \to 0^+$ uniformly in that part of M where (7.2) no longer holds.

Proof. In the region T_1 we have that z is given by (7.2). Thus the substitution of (7.1) into (4.2) will yield

$$\frac{1}{\varepsilon}\ e^{-g/\varepsilon}\ [g_x^2 - (ag_x + bg_y)] \sum_{n=o}^{N} \varepsilon^n\ h_n$$

$$+ e^{-g/\varepsilon} \sum_{n=o}^{N} \varepsilon^n\ [(a - 2g_x)\ h_{n,x} - bh_{n,y} + (c - g_{xx})\ h_n]$$

$$+ e^{-g/\varepsilon} \sum_{n=o}^{N} \varepsilon^{n+1}\ h_{n,xx} + \varepsilon^{N+1}[\varepsilon Z_{xx} + aZ_x - bZ_y + cZ] = 0\ .$$

As before we obtain $g(x,y)$ as the solution to problem P_4 and $h_o(x,y)$ as the solution to problem P_5. For $h_n(x,y)$ we examine the coefficient of ε^n to obtain $(n = 1,2,...,N)$

$$P'_n : \quad \text{PDE} \quad (a - 2g_x)h_{n,x} - bh_{n,y} + (c - g_{xx})h_n = -h_{n-1,xx}$$
$$\text{in } M \text{ near } J^*$$

$$\text{IC} \quad h_n = 0 \quad \text{on } J^*\ .$$

Now as a, b, $x(s)$, $y(s)$ $\varepsilon\ C^{4+2N}$, we have that $g\ \varepsilon\ C^{3+2N}$. Again, the characteristics in the xy-plane for P_4 and P'_n $(n = 0,1,...,N)$ are the same. Also, $h_n\ \varepsilon\ C^{1+2(N-n)}$. The partial differential equation solved by Z is

$$\varepsilon Z_{xx} + aZ_x - bZ_y + cZ = \begin{cases} - h_{N,xx}\ e^{-(g/\varepsilon)} & \text{in } T_1 \\ - \dfrac{1}{\varepsilon^{N+1}}\ [\varepsilon z_{xx} + az_x - bz_y + cz] & \\ \qquad = 0\left(\dfrac{e^{-(\delta/\varepsilon)}}{\varepsilon^{N+1}} \right) \text{ as } \varepsilon \to 0^+ & \text{in } T_2 \\ 0 \quad \text{in } (D + \partial D) - (T_1 + T_2). & \end{cases}$$

Thus the problem for Z may be expressed as

PDE $\quad \varepsilon Z_{xx} + aZ_x - bZ_y + cZ = O(1)$ uniformly in $(D + \partial D)$

as $\varepsilon \to 0^+$

IC $\quad Z = 0 \quad$ on $\Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3$.

By the maximum principle of lemma 3.3, $Z(x,y;\varepsilon) = O(1)$ as $\varepsilon \to 0^+$ uniformly in $D + \partial D$.

8. <u>A Refinement of the General Case of §5</u>

The modification in this case is essentially the same as given in §9 for the elliptic case. We shall have that

$$u(x,y;\varepsilon) = U_o(x,y) + \sum_{n=1}^{N} \varepsilon^n U_n(x,y) + \tilde{u}_2(x,y;\varepsilon) + \tilde{u}_3(x,y;\varepsilon) + \varepsilon^N w(x,y;\varepsilon)$$

where $\tilde{u}_2$ and $\tilde{u}_3$ are defined as in §5 but with a remainder term $\varepsilon^{N+1}Z$ obtained as in §7.

As before, U_o solves the reduced problem P_2 (and is thus the same as U in §5) while $U_n(x,y)$ solves problem P_n'' $(n = 1,2,\ldots,N)$

PDE $\quad aU_{n,x} - bU_{n,y} + cU_n = - U_{n-1,xx} \quad$ in $(M + \partial M)$

IC $\quad U_n = 0 \quad$ on I.

The function $g^{(\alpha)}(x,y)$ associated with $\tilde{u}_\alpha$ $(\alpha = 1,2)$ is again the same as in §4, and $h_o^{(\alpha)}(x,y)$ agrees with $h^{(\alpha)}(x,y)$ of §4. The function $h_n^{(\alpha)}(x,y)$ solves the problem P_n''' $(n = 1,2,\ldots,N)$

PDE $\quad (a - 2g_x)h_{n,x}^{(\alpha)} - bh_{n,y}^{(\alpha)} + (c - g_{xx})h_n^{(\alpha)} = - h_{n-1,xx}^{(\alpha)} \quad$ in M near $J_{(\alpha)}^*$

IC $\quad h_n^{(\alpha)} = - U_n \quad \text{on} \quad J^*_{(\alpha)}$.

For the function $Z^{(\alpha)}$ we have the problem

$$\text{PDE} \quad \varepsilon Z_{xx}^{(\alpha)} + aZ_x^{(\alpha)} - bZ_y^{(\alpha)} + cZ^{(\alpha)} = \begin{cases} -h_{N,xx}^{(\alpha)} e^{-(g/\varepsilon)} & \text{in } T_1 \\ -\dfrac{1}{\varepsilon^{N+1}}[\varepsilon z_{xx}^{(\alpha)} + az_x^{(\alpha)} - bz_y^{(\alpha)} + cz^{(\alpha)}] & \text{in } T_2 \\ 0 \quad \text{in } (M + \partial M) - (T_1 + T_2) \end{cases}$$

IC $\quad Z = 0 \quad \text{on} \quad \Gamma_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3$.

By choosing δ_1 as given in §4 sufficiently small, we can insure that the R.H.S. of the PDE is $0(1)$ so that, by the maximum principle of lemma 3.3, $Z = 0(1)$ in M .

As u, $\sum_{n=o}^{N} \varepsilon^n U_n$, $\tilde{u}_2$, $\tilde{u}_3$ are all defined, so also is w as the solution of

$$\text{PDE} \quad \varepsilon w_{xx} + aw_x - bw_y + cw = - \varepsilon U_{N,xx} \quad \text{in} \quad (M + \partial M)$$

$$\text{IC} \quad \begin{cases} w = 0 \quad \text{on} \quad I + J^* \\ w \text{ in general is non-zero on } \partial M - (I + J^*). \end{cases}$$

Since this is the same as that of §6, we once again obtain that $w = 0(\varepsilon)$ uniformly in M as $\varepsilon \to 0^+$.

9. An Extension of the Special Case of §4

We wish to extend our problem to the case where

$$a(x,y;\varepsilon) = \sum_{m=o}^{M} \varepsilon^m a_m(x,y)$$

and b, c, d are similarly defined, as are the boundary conditions for u on ∂D.

The problem for $u(x,y;\varepsilon)$ is

$$\text{PDE} \qquad \varepsilon u_{xx} + a\, u_x - b\, u_y + cu = 0 \qquad (x,y) \in D \tag{9.1}$$

$$\text{IC} \qquad u = \begin{cases} \theta(s;\varepsilon) & (x,y) \in J^*) \\ 0 & (x,y) \in (\partial D - J^*) \end{cases}$$

where we assume that

(i) $\theta(s;\varepsilon) = \sum_{n=o}^{N} \varepsilon^n \theta_n(s)$ with $\theta_n \in C^{1+2(N-n)}(J^*)$

(ii) $\theta \equiv 0$ over a small segment at each end of J^*.

Its associated reduced problem has the solution $U(x,y;\varepsilon) \equiv 0$ in $D + \partial D$.

We shall construct $u(x,y;\varepsilon)$ of the form

$$u(x,y;\varepsilon) = z(x,y;\varepsilon) + \varepsilon^{N+1} Z(x,y;\varepsilon) \tag{9.2}$$

where

$$z = e^{-(g(x,y)/\varepsilon)} \sum_{n=o}^{N} \varepsilon^n h_n(x,y) \quad \text{with} \quad N \geq M$$

and

$$Z = \sum_{m=o}^{M} \varepsilon^m Z_m (x,y;\varepsilon).$$

Substituting (9.2) into (9.1) yields the following results in the region T_1 :

$g(x,y)$ solves

PDE $\quad g_x^2 - a_o g_x + b_o g_y = 0$

IC $\quad g = 0$ on J^* and $g > 0$ in M near J^* .

Thus g is the same as in problem P_4. h_n solves

PDE
$$(a_o - 2g_x)h_{n,x} - b_o h_{n,y} + (c_o - g_{xx} - a_1 g_x + b_1 g_y)h_n$$
$$= h_{n-1,xx} - \sum_{k=1}^{n} (a_k h_{n-k,x} - b_k h_{n-k,y} + c_k h_{n-k})$$
$$+ \sum_{k=1}^{n} (a_{k+1} g_x - b_{k+1} g_y) h_{n-k} .$$

IC $\quad h_n = \theta_n$ on J^* .

And the partial differential equations:

$$\varepsilon Z_{o,xx} + a_o Z_{o,x} - b_o Z_{o,y} + c_o Z_o = e^{-(g/\varepsilon)}\left[\sum_{k=1}^{M-1} (a_{k+1} g_x - b_{k+1} g_y) h_{N+1-k}\right.$$
$$\left. - \sum_{k=1}^{M} (a_k h_{N+1-k,x} + b_k h_{N+1-k,y} + c_k h_{N+1-k}) - h_{N,xx}\right] \tag{9.3}$$

$$\varepsilon Z_{m,xx} + a_o Z_{m,x} - b_o Z_{m,y} + c_o Z_m = e^{-(g/\varepsilon)}\left[\sum_{k=1}^{M-1} (a_{k+1} g_x - b_{k+1} g_y) h_{N+1+m-k}\right.$$
$$\left. - \sum_{k=1}^{M} (a_k h_{N+1+m-k,x} - b_k h_{N+1+m-k,y} + c_k h_{N+1+m-k})\right]$$
$$- \sum_{k=1}^{M} (a_k Z_{m-k,x} - b_k Z_{m-k,y} + c_k Z_{m-k}) \tag{9.4}$$

$$\varepsilon Z_{M,xx} + a_o Z_{M,x} - b_o Z_{M,y} + c_o Z_M = - \sum_{k=1}^{M} \{a_k Z_{M-k,x} - b_k Z_{M-k,y} + c_k Z_{M-k}\}$$

$$- \sum_{m=1}^{M} \varepsilon^m \sum_{k=m}^{M} \{a_k Z_{M+m-k,x} - b_k Z_{M+m-k,y} + C_k Z_{M+m-k}\} \tag{9.5}$$

For the region T_2, we define

$$z(\sigma,\tau;\varepsilon) = \sum_{n=o}^{N} \varepsilon^n z_n(\sigma,\tau;\varepsilon)$$

where $z_n(\sigma,\tau;\varepsilon) = (1 - \frac{\tau}{\tau_1})^3 \sum_{m=o}^{2} \tau^m z_{mn}(\sigma;\varepsilon)$ continues the solution $e^{-(g/\varepsilon)} h_n$ from T_1 into T_2 such that $z \in C^2(T_1 + T_2)$.

The remainder Z_o satisfies the PDE (9.3) in T_1 (9.6) below in T_2, and (9.7) in $D - (T_1 + T_2)$ where

$$\varepsilon Z_{o,xx} + a_o Z_{o,x} - b_o Z_{o,y} + C_o Z_o = \frac{-1}{\varepsilon^{N+1}} \{\varepsilon z_{xx} + a z_x - b z_y + c z\}$$

$$= 0 \left(\frac{e^{-(\delta_1/\varepsilon)}}{\varepsilon^{N+1}} \right) \text{ as } \varepsilon \to 0^+ , \tag{9.6}$$

$$\varepsilon Z_{o,xx} + a_o Z_{o,x} - b_o Z_{o,y} + c_o Z_o = 0 \tag{9.7}$$

With the initial condition $Z_o = 0$ on Γ, we have that $Z_o = 0(1)$ as $\varepsilon \to 0^+$ uniformly in D. In the similar fashion, we have that Z_m $(m = 1,2,\ldots,M-1)$ satisfies (9.4) in T_1 while in $D - T_1$ we have

$$\varepsilon Z_{m,xx} + a_o Z_{m,x} - b_o Z_{m,y} + c_o Z_m = - \sum_{k=1}^{M} (a_k Z_{m-k,x} - b_k Z_{m-k,y} + C_k Z_{m-k}) .$$

Since the initial conditions are $Z_m = 0$ on Γ, we have, by induction, that $Z_m = 0(1)$ as $\varepsilon \to 0^+$ uniformly in D. Finally Z_M satisfies (9.5) in D so that, with the initial condition $Z_M = 0$ on Γ we have that $Z_M = 0(1)$ as $\varepsilon \to 0^+$ uniformly in D. Thus we conclude that the remainder term Z is 0(1) uniformly in D as $\varepsilon \to 0^+$.

As before, we wish a, b, c, x(s), y(s) $\varepsilon\ C^{4+2N}$ so that $g\ \varepsilon\ C^{3+2N}$, $h_n\ \varepsilon\ C^{1+2(N-n)}$ for $n = 0,1,\ldots,N$ and $Z\ \varepsilon\ C^2$ where $\theta_n\ \varepsilon\ C^{3+2(N-n)}$.

10. An Extension of the General Case of §5

As in §9, we wish to have a, b, c, d expressed as polynomials of order M in terms of ε. We shall construct a solution of form

$$u(x,y;\varepsilon) = U_o(x,y) + \sum_{n=1}^{N} \varepsilon^n U_n(x,y) + \tilde{u}_2(x,y;\varepsilon) + \tilde{u}_3(x,y;\varepsilon) + \varepsilon^N w(x,y;\varepsilon).$$

The discussion is almost the same as in §8, except U_n solves $(n = 1,2,\ldots,N)$

$$\text{PDE} \quad a_o U_{n,x} = b_o U_{n,y} + C_o U_n = - U_{n-1,xx} = \sum_{k=1}^{n} (a_k U_{n-k,x} - b_k U_{n-k,y} + c_k U_{n-k}) \quad \text{in} \quad (M + \partial M).$$

IC $U_n = 0$ on J^*.

For h_n, the initial condition becomes $h_n = \theta_n - U_n$ on J^*. Throughout the entire region D we add to the R.H.S. of the equation

in (9.3) or (9.4) the term

$$\sum_{k=m}^{M} (a_k U_{N+m-k,x} - b_k U_{N+m-k,y} + c_k U_{N+m-k}) \qquad (m = 1,2,\ldots,M)$$

Note that this does not change the order of Z_m $(m = 0,1,\ldots,M-1)$. Since the problem solved by w in §8 is unchanged, we have once again $w = O(\varepsilon)$ uniformly in M as $\varepsilon \to 0^+$.

CHAPTER IV

WASOW'S METHOD FOR DEALING WITH THE ELLIPTIC 2^{nd} ORDER

$$\text{PDE} \quad \varepsilon\Delta u - \frac{\partial u}{\partial x} = f(x,y)$$

1. The Statement of the Theorem

In [10] Wasow proved the following theorem and corollaries.

Theorem. Let D be a finite open plane domain of finite connectivity bounded by a finite number of arcs with continuously turning tangents. Let ∂D denote the boundary of D and let Γ be the right-hand boundary of D (see Fig.4.1); and $\Gamma : x = q(y) \ \varepsilon \ C^1_p([a,b])$. $P(x,y)$ is a point in the open domain D; $P_o(x_o,y)$ is the point on Γ through which the abscissa through P passes. (Note that P may not lie on the left-hand boundary of D). $f(x,y) \ \varepsilon \ C(D + \partial D)$ and $f(x,y) \ \varepsilon \ C^1(D)$. Consider

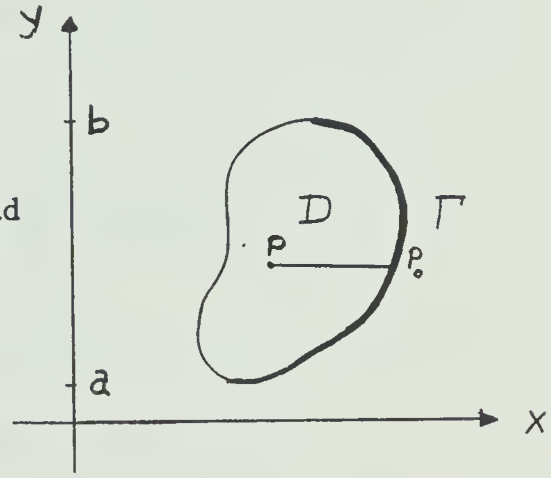

Figure 4.1

Problem I

$$\text{PDE} \quad \frac{\partial w}{\partial x} = f(x,y) \qquad (x,y) \ \varepsilon \ D \tag{1.1}$$

$$\text{IC} \quad w(x,y) = 0 \qquad (x,y) \ \varepsilon \ \Gamma \tag{1.2}$$

with solution

$$w(x,y) = \int_{x_o}^{x} f(\xi,y)d\xi \ ; \tag{1.3}$$

Problem II

$$\text{PDE} \quad \varepsilon\Delta u + \frac{\partial u}{\partial x} = f(x,y) \qquad (x,y) \ \varepsilon \ D, \quad \varepsilon > 0 \tag{1.4}$$

$$\text{BC} \qquad u(x,y;\varepsilon) = 0 \qquad (x,y) \in \partial D \tag{1.5}$$

with solution $u(x,y;\varepsilon)$.

Then

$$\lim_{\varepsilon \to 0} u(x,y;\varepsilon) = w(x,y) \tag{1.6}$$

uniformly in a neighborhood of the line segment $\overline{PP_o}$.

Corollary 1. If (1.2) is replaced by $w(x,y) = \phi(x,y)$ for $(x,y) \in \Gamma$, where $\phi \in C(\Gamma)$ and (1.5) is replaced by

$$u(x,y;\varepsilon) = \begin{cases} \phi(x,y) & (x,y) \in \Gamma \\ \psi(x,y) & (x,y) \in (\partial D - \Gamma) \end{cases},$$

where ψ is so defined that $u \in C(\partial D)$, then

$$\lim_{\varepsilon \to 0} u(x,y;\varepsilon) = \phi(x_o,y) + \int_{x_o}^{x} f(\xi,y)d\xi$$

uniformly in a neighborhood of the line segment $\overline{PP_o}$.

Corollary 2. If $\varepsilon < 0$, then Γ is replaced by $\partial D - \Gamma$.

2. An Outline of the Technique that Wasow Used to Prove the Theorem

2.1. First the proof of the theorem is given for a special case characterized by the following additional restrictions:

(A) Let the boundary ∂D of D be composed of two open arcs Γ_0 and Γ_1 defined by the functions $x = q_0(y)$ and $x = q_1(y)$ for $a \leq y \leq b$ respectively, and of two closed line

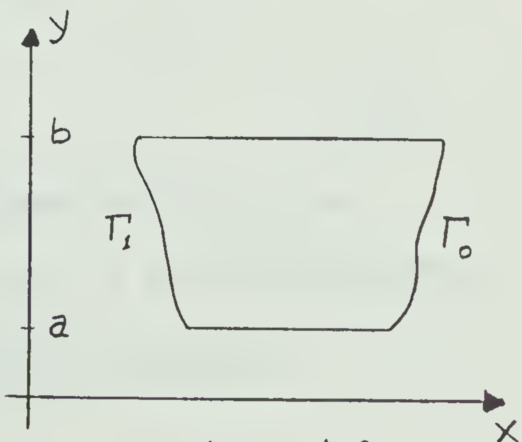

Figure 4.2

segments $y = a$ and $y = b$, where it is assumed that

$q_1 < q_0$ for $a \leq y \leq b$ (see Fig. 4.2).

(B) $q_0(y) \in C^3(\{y | a \leq y \leq b\})$.

(C) $f(x,y) \in C^3(D + \partial D)$.

(D) $w(x,y) = \int_{q_0(y)}^{x} f(\xi,y)d\xi$ vanishes on ∂D.

Define $E(x,y;\varepsilon)$ by $u(x,y;\varepsilon) = w(x,y) + \varepsilon \cdot E(x,y;\varepsilon)$. Then $E(x,y;\varepsilon)$ solves

$$\text{PDE} \qquad \varepsilon \Delta E + \frac{\partial E}{\partial x} = - \Delta W \qquad (x,y) \in D$$

$$\text{BC} \qquad E = 0 \qquad (x,y) \in \partial D.$$

To show (1.6), we need only show that $E(x,y;\varepsilon)$ is bounded in $D + \partial D$.

If $G(x,y;\xi,\eta;\varepsilon)$ is the Green's function for the operator $\varepsilon \Delta u + \frac{\partial u}{\partial x}$ over the domain D, then E can be expressed as

$$E(x,y;\varepsilon) = \int\int_D G(x,y;\xi,\eta;\varepsilon)\Delta w \, d\xi d\eta \, .$$

It is shown that

$$G(x,y;\xi,\eta;\varepsilon) = \frac{1}{\varepsilon} K(x,y;\xi,\eta;\varepsilon) \, e^{\frac{\xi - x}{2\varepsilon}} \, , \qquad (2.11)$$

where $K(x,y;\xi,\eta;\varepsilon)$ is the Green's function for the operator $\varepsilon^2 \Delta u - \frac{1}{4} u$ over the domain D. By (i) the fact that

$$0 < K < \gamma(r;\varepsilon) \, , \qquad (2.12)$$

where $\gamma = \frac{i}{4} H_o^{(1)}\left(\frac{ir}{2\varepsilon}\right)$, a Hankel function of the first kind of order zero, is a fundamental solution for $\Delta u - \frac{1}{4\varepsilon^2} u$[1] and $r = \sqrt{(x-\xi)^2+(y-\eta)^2}$, and (ii) the asymptotic formulae of the modified Bessel and Hankel

---

(1) See [9] p. 566.

functions, we can show that $E(x,y;\varepsilon)$ is bounded in D.

2.2. Removal of Assumption (D). Any domain satisfying assumption (A) will be called normal. Let D' by any closed subdomain of $D + \Gamma_0$. Then D' can be embedded in a closed normal subdomain D^* of $D + \Gamma_0$ such that (i) the right-hand boundary arc of D^* is part of Γ_0, (ii) D' has a minimum distance $r_o > 0$ from $D - D^*$ (see Fig. 4.3). Let $\rho(x,y) \in C^4(D + \partial D)$ be a function such that

(i) $\rho = 1 \quad (x,y) \in D^*$

(ii) $\rho = 0 \quad (x,y) \in (\partial D - \Gamma_0)$.

Then under the conditions (A), (B) and (C) in §2.1 and

$$w(x,y) = \int_{q_0(y)}^{x} f(\xi,y)d\xi \ ,$$

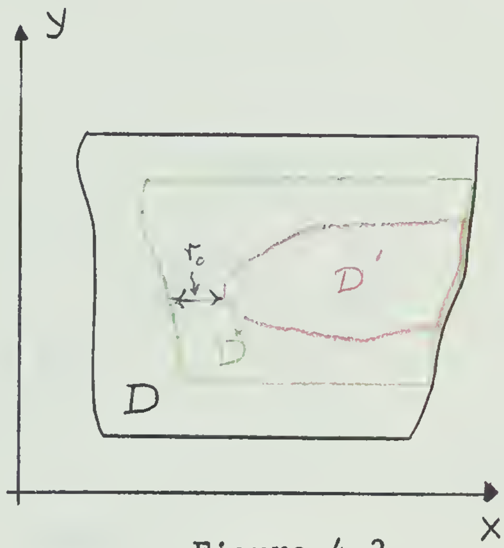

Figure 4.3

$\rho(xy) \cdot w(x,y)$ satisfies condition (D). And also if $f^*(x,y) = \frac{\partial}{\partial x}(\rho w)$ and $u^*(x,y;\varepsilon)$ solves

$$\text{PDE} \qquad \varepsilon\Delta u^* + \frac{\partial u^*}{\partial x} = f^* \qquad (x,y) \in D$$

$$\text{BC} \qquad u^* = 0 \qquad (x,y) \in \partial D,$$

then by the result in §2.1 we have that

$$\lim_{\varepsilon \to 0} u^* = \rho w \quad \text{uniformly in} \quad (D + \partial D).$$

Let $H(x,y;\varepsilon) = u(x,y;\varepsilon) - u^*(x,y;\varepsilon)$. Then H solves

$$\text{PDE} \qquad \varepsilon\Delta H + \frac{\partial H}{\partial x} = f - f^* \qquad (x,y) \in D$$

$$\text{BC} \qquad H = 0 \qquad (x,y) \in \partial D \ .$$

To show that (1.6) in §1 holds uniformly in D', we must show that

$$\lim_{\varepsilon \to 0} H(x,y;\varepsilon) = 0 \qquad (x,y) \in D'.$$

Again

$$H(x,y;\varepsilon) = \iint_{D-D^*} (f - f^*)\, G(x,y;\xi,\eta;\varepsilon)\, d\xi d\eta. \tag{2.21}$$

By (2.11) and (2.12) in §2.1 and the fact that

$$\gamma(r;\varepsilon) \sim \frac{1}{2}\sqrt{\frac{\varepsilon}{\pi r}}\; e^{-\frac{r}{2\varepsilon}},$$

we have

$$0 < G < \frac{1}{\varepsilon} \cdot \gamma \cdot e^{\frac{r\cos\theta}{2\varepsilon}}$$

$$< \frac{1}{2}\,\frac{K}{\sqrt{\pi\varepsilon r}}\; e^{-\frac{r}{2\varepsilon}(1-\cos\theta)}$$

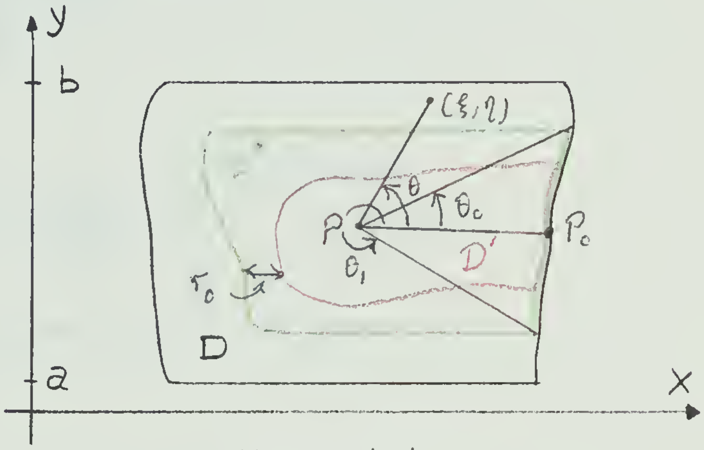

Figure 4.4

for $\frac{r}{2\varepsilon} > \tau_o$, where K depends on the choice of τ_o. Since $r > r_o$, we need only require that $\varepsilon < \frac{r_o}{2\tau_o}$. Since $(x,y) \in D'$ and $(\xi,\eta) \in (D - D^*)$, then

$$\begin{cases} r_o \le r \le R, \\ \theta_o \le \theta \le \theta_1, \end{cases}$$

where R is greater than the diameter of D. Thus $-1 \le \cos\theta < 1$.

Let $\theta_2 = \min\limits_{\substack{(x,y)\,\in\, D' \\ (\xi,\eta)\,\in\,(D-D^*)}} \left\{ \theta_0\,,\ 2\pi - \theta_1 \right\}$ (see Figure 4.4) ,

Then

$$0 < G(x,y;\xi,\eta;\varepsilon) < \frac{1}{2} \cdot \frac{K}{\sqrt{\pi\varepsilon r}}\; e^{-\frac{r}{2\varepsilon}(1-\cos\theta_2)} \tag{2.22}$$

(2.22) approaches to zero as $\varepsilon \to 0^+$ for $(x,y) \in D'$ and

$(\xi,\eta) \varepsilon (D - D^*)$. From this and (2.21), we have that

$$\lim_{\varepsilon \to 0^+} H(x,y;\varepsilon) = 0 \qquad (x,y) \varepsilon D'.$$

At this point Corollary 2 (i.e. if $\varepsilon < 0$, then Γ_0 is replaced by Γ_1) is proved. If $\varepsilon < 0$, then we replace $\gamma(r;\varepsilon)$ by $\gamma(r;-\varepsilon)$. Then asymptotically we have

$$\gamma(r;-\varepsilon) \sim \frac{1}{2}\sqrt{\frac{-\varepsilon}{2r}}\; e^{\frac{r}{2\varepsilon}} ,$$

so that

$$0 < G(x,y;\xi,\eta;\varepsilon) < \frac{1}{2} \cdot \frac{K}{\sqrt{-\pi\varepsilon r}}\; e^{\frac{r}{2\varepsilon}(1+\cos\theta)}$$

where $2\pi - \theta_1 \leq \theta \leq \theta_0$, so that $-\alpha \leq \cos\theta \leq 1$ for some α, $0 < \alpha < 1$.

2.3. <u>Removal of Assumption (C)</u>. Let $\{f_\nu(x,y)\}\big|_{\nu=1}^{\infty}$ be a sequence of functions with $f_\nu \varepsilon C^3(D + \partial D)$, which converges uniformly to $f(x,y)$ where $f \varepsilon C^1(D)$ and $f \varepsilon C(D + \partial D)$. Then we show that

(i) $w_\nu(x,y) \to w(x,y)$ uniformly in $(D + \partial D)$ as $\nu \to \infty$,

(ii) $u_\nu(x,y;\varepsilon) \to u(x,y;\varepsilon)$ uniformly in $(D + \partial D)$ as $\nu \to \infty$, and

(iii) $\lim_{\varepsilon \to 0} u_\nu(x,y;\varepsilon) = w_\nu(x,y)$ uniformly in D' as defined in §2.2,

where $w_\nu(x,y)$ satisfies

$$\text{PDE} \qquad \frac{\partial w_\nu}{\partial x} = f_\nu \qquad (x,y) \varepsilon D$$

$$\text{IC} \qquad w_\nu = 0 \qquad (x,y) \varepsilon \Gamma_0$$

and $u_\nu(x,y;\varepsilon)$ solves

$$\text{PDE} \qquad \varepsilon\Delta u_\nu + \frac{\partial u_\nu}{\partial x} = f_\nu \qquad (x,y) \in D$$

$$\text{BC} \qquad u_\nu = 0 \qquad (x,y) \in \partial D.$$

From the results (i), (ii), and (iii) above, the relation (1.6) holds in D'.

2.4. Removal of Assumption (B). Here condition (B) is replaced by the assumption that $q_0(y) \in C_P^1$. Let $\Gamma_0^{(\mu)}$ be the arc (replacing Γ_0) given by

$$x = q_0^{(\mu)}(y) \ , \quad a \le y \le b,$$

such that

$$\text{(i)} \quad q_0^{(\mu)}(y) \in C^3,$$

$$\text{(ii)} \quad q_0(y) < q_0^{(\mu)} < q_0(y) + \frac{1}{\mu} \ .$$

$D^{(\mu)}$ is the region bounded by the arcs $\Gamma_0^{(\mu)}$, Γ_1, $y = a$ and $y = b$. $q_2(y)$ is a function defined on $a \le y \le b$ such that

(i) $q_2(y) \in C^1$,

(ii) $q_1 < q_2 < q_0$, $a \le y \le b$.

Define a mapping
$$x' = x + \frac{x - q_2(y)}{m} \cdot \frac{1}{\mu}$$
$$y' = y$$

where m is a constant such that

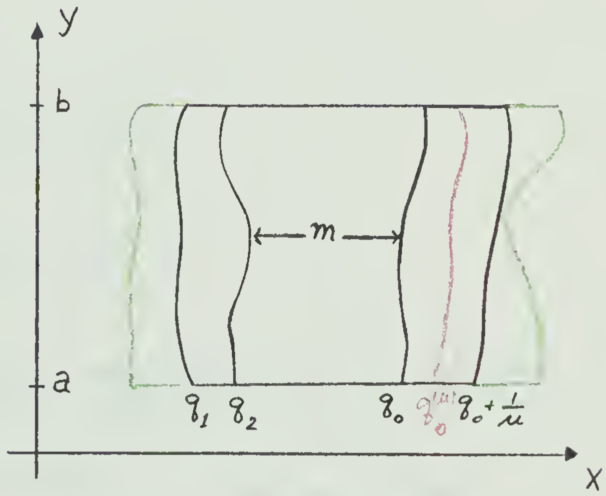

Figure 4.5

$$0 < m < \min_{(x,y)\in\Gamma_0} \{x - q_2(y)\} = \min_{a\le y\le b} \{q_o(y) - q_2(y)\},$$

as shown in Fig. 4.5.

By this mapping $(x,y) \varepsilon (D + \partial D)$ is mapped into (x',y') in the region bounded by the green lines as $x - q_2(y) \geq m$. Now let

$$
\begin{aligned}
f_\mu(x',y') &= f(x,y) \\
&= f\left(x' - \frac{x' - q_2(y)}{1 + m\mu}, y'\right).
\end{aligned}
$$

Then $f_\mu \ \varepsilon \ C^1(D^{(\mu)})$ and $f_\mu(x,y) \to f(x,y)$ uniformly as $\mu \to \infty$ in D. Then we show that

(i) $w_\mu \to w$ uniformly in $D + \partial D$ as $\mu \to \infty$,

(ii) $u_\mu \to u$ uniformly in $D + \partial D$ as $\mu \to \infty$, and

(iii) $\lim\limits_{\varepsilon \to 0} u_\mu = w_\mu$ uniformly in any subdivision of $D^{(\mu)} + \Gamma_0^{(\mu)}$,

where $w_\mu(x,y) = \int_{q_0^{(\mu)}(y)}^{x} f_\mu(\xi,y)d\xi$ solves

$$
\begin{array}{lll}
\text{PDE} & \dfrac{\partial w_\mu}{\partial x} = f_\mu & (x,y) \ \varepsilon \ D^{(\mu)} \\[2ex]
\text{IC} & w_\mu = 0 & (x,y) \ \varepsilon \ \partial D^{(\mu)}
\end{array}
$$

and $u_\mu(x,y;\varepsilon)$ solves

$$
\begin{array}{lll}
\text{PDE} & \varepsilon\Delta u_\mu + \dfrac{\partial u_\mu}{\partial x} = f_\mu & (x,y) \ \varepsilon \ D^{(\mu)} \\[2ex]
\text{BC} & u_\mu = 0 & (x,y) \ \varepsilon \ \partial D^{(\mu)} .
\end{array}
$$

By the above results $\lim\limits_{\varepsilon \to 0} u(x,y;\varepsilon) = w(x,y)$ uniformly in D follows.

2.5. Removal of Assumption (A). In this section D is no longer necessarily normal. Let D' be a subdomain of $D + \partial D$ such that D' can be enclosed by a normal subdomain D^* with its right boundary arc

Γ_0^* contained in Γ_0. Let $u(x,y;\varepsilon)$ be the solution of Problem II in §1 and $u(x,y;\varepsilon) = \Phi(x,y;\varepsilon)$ for $(x,y) \in \partial D^*$. In D^*, $u(x,y;\varepsilon)$ can be written as

$$u(x,y;\varepsilon) = u_1(x,y;\varepsilon) + u_2(x,y;\varepsilon) ,$$

where $u_1(x,y;\varepsilon)$ satisfies

$$\text{PDE} \qquad \varepsilon\Delta u_1 + \frac{\partial u_1}{\partial x} = f(x,y) \qquad (x,y) \in D^*$$

$$\text{BC} \qquad u_1 = 0 \qquad (x,y) \in \partial D^*$$

and $u_2(x,y;\varepsilon)$ satisfies

$$\text{PDE} \qquad \varepsilon\Delta u_2 + \frac{\partial u_2}{\partial x} = 0 \qquad (x,y) \in D^*$$

$$\text{BC} \qquad u_2 = \Phi \qquad (x,y) \in \partial D^* .$$

Since D^* is normal domain, we have by §2.4 that

$$\lim_{\varepsilon \to 0} u_1 = w$$

uniformly in D'. The fact that (1.6) holds uniformly in D' now follows from the proof that $\lim\limits_{\varepsilon \to 0} u_2 = 0$ uniformly in D'.

3. Extension

The preceding results can be immediately extended to the more general partial differential equation

$$L[u] = \varepsilon\Delta u + a\,\frac{\partial u}{\partial x} + b\,\frac{\partial u}{\partial y} = f$$

provided a and b are conjugate harmonic functions not having a

common zero in $D + \partial D$, for then there exists a conformal mapping taking $L[u]$ into

$$(a^2 + b^2)\ \{\varepsilon\Delta' u + \frac{\partial u}{\partial x'}\}\quad .$$

The above statement can be shown as follows: Let $w = x + iy$, $z = x' + iy'$ where $z = f(w)$ is an analytic function. Then we have that (1) Cauchy-Riemann equations $x'_x = y'_y$, $x'_y = -y'_x$, (3.1)

(2) $\Delta x' = \Delta y' = 0$. (3.2)

Now by the chain-rule of differentiation and (3.1), (3.2), we have

$$\varepsilon\Delta u + au_x + bu_y = \varepsilon[(x'_x)^2 + (x'_y)^2]\ (u_{x'x'} + u_{y'y'}) \\ + (ax'_x + bx'_y)\ u_{x'} + (ay'_x + by'_y)u_{y'}\ . \tag{3.3}$$

If $z = f(w)$ and $\frac{dz}{dw} = a + ib$, then a, b are a pair of conjugate harmonic functions:

$$x'_x = a = y'_y\ ,\quad x'_y = b = -\ y'_x\ .$$

We will have a conformal mapping in $D + \partial D$ as long as $\frac{dz}{dw} \neq 0$ at any point in $D + \partial D$. Then we have from (3.3)

$$\begin{aligned}\varepsilon\Delta u + au_x + bu_y &= \varepsilon(a^2 + b^2)\Delta' u + (a^2 + b^2)\ u_{x'} \\ &= (a^2 + b^2)\ \{\varepsilon\Delta' u + u_{x'}\}\quad ,\end{aligned}$$

where $\Delta' u = u_{x'x'} + u_{y'y'}$.

CHAPTER V

A NOTE ON THE ASYMPTOTIC SOLUTION FOR HEAT EQUATION

In the present chapter we apply the technique of Wasow [10] to the heat equation in one space variable. Everything stated below can be carried over to the case of several space variables.

1. Statement of the Theorem

Consider an open domain D bounded below by the straight line $t = 0$ and laterally by AC and CD where AC and BD do not include the end-points A and B (see Figure 5.1). Consider

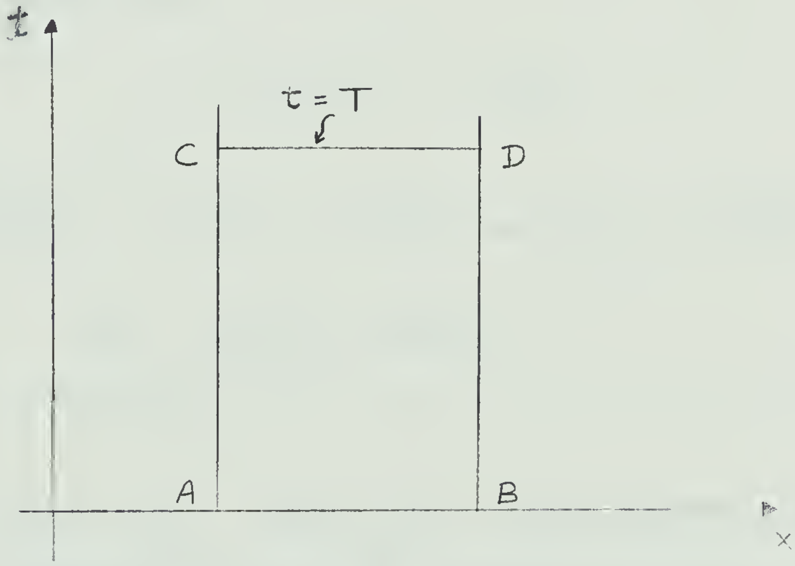

Figure 5.1

the partial differential equation

$$\varepsilon U_{xx} - U_t = f(x,t) \qquad (x,t) \in D \tag{1.1}$$

with initial condition

$$u = \phi(x) \qquad x \in AB \tag{1.2}$$

and the boundary conditions[(1)]

$$u = \psi_1(x,t) = w(x,t) \qquad (x,t) \in AC \tag{1.3}$$

$$u = \psi_2(x,t) = w(x,t) \qquad (x,t) \in BD \tag{1.4}$$

where $w(x,t)$ is given by

$$w(x,t) = \phi(x) - \int_0^t f(x,\tau)\, d\tau$$

and satisfies the reduced equation (i.e. put $\varepsilon = 0$ in (1.1))

$$w_t = -f(x,t) \qquad (x,t) \in D$$

and the initial condition

$$w = \phi(x) \qquad x \in AB \ .$$

If $f(x,t) \in C(D+\partial D)$, $f(x,t) \in C^1(D)$ and $\phi(x) \in C^1(AB)$, then

$$\lim_{\varepsilon \to 0^+} U(x,t;\varepsilon) = w(x,t) \tag{1.5}$$

uniformly for $(x,t) \in D_1$, where D_1 is the closed domain bounded below by AB , laterally by AC, BD and above by $CD : t = T$.

---

(1) Note the restriction on the boundary conditions. This will be clarified later.

2. Special Case

Now the proof of the theorem will first be given under the following additional restrictions:

(a) $f(x,t) \varepsilon C^2(D+\partial D)$, $f(x,t) \varepsilon C^3(D)$

(b) $\phi(x) \varepsilon C^3(AB)$.

Assume

$$u(x,y;\varepsilon) = w(x,t) + \varepsilon E(x,t;\varepsilon) , \tag{1.6}$$

where $E(x,y;\varepsilon)$ solves

$$\varepsilon E_{xx} - E_t = -w_{xx} \qquad (x,t) \varepsilon D_1 \tag{1.7}$$

with the initial condition

$$E = 0 \qquad (x,t) \varepsilon AB \tag{1.8}$$

and the boundary conditions

$$E = 0 \qquad (x,t) \varepsilon AC \text{ and } BD . \tag{1.9}$$

There exists a unique solution of (1.7), (1.8) and (1.9) of the form(1)

$$E(x,t;\varepsilon) = \int_{D_1}\int G(x,t;\xi,\tau;\varepsilon) w_{\xi\xi} \, d\xi \, d\tau , \tag{1.10}$$

where $G = G_o - v$ and

$$G_o(x,t;\xi,\tau;\varepsilon) = \frac{1}{2\sqrt{\pi\varepsilon(t-\tau)}} e^{-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}}$$

---

(1) The proof can be found in [9], pp. 521-525; 209-210.

is the fundamental solution of the heat equation

$$\varepsilon U_{xx} - U_t = 0 \qquad (x,t) \in D_1 .$$

The term $v(\xi,\tau;x,t;\varepsilon)$, when considered as a function of ξ and τ containing the parameters x,t,ε, satisfies

$$\varepsilon V_{\xi\xi} + V_\tau = 0 \quad \text{for} \quad \tau < t \qquad (x,t) \in D_1$$

with the initial condition

$$V = 0 \quad \text{on} \quad CD : t = \tau$$

$$V = G_o(x,t;\xi,\tau;\varepsilon) \quad \text{on} \quad AC \text{ and } BD$$

and is of class C^1 in the interior of D ; or $v(x,t;\xi,\tau;\varepsilon)$ satisfies, with respect to x and t ,

$$\varepsilon V_{xx} - V_t = 0 \quad \text{for} \quad t > \tau$$

and

$$V = 0 \qquad \text{for} \quad t = \tau \tag{1.11}$$

$$V = G_o \quad \text{on} \quad AC \text{ and } BD .$$

V and G depend only on the difference $t - \tau$. Note that as $t \to \tau + 0$, with fixed x not equal to ξ , the boundary values (1.11) are continuous at the corners of the region. On observing that the boundary values are non-negative, we have by the maximum principle for the parabolic equation(1) that $V \geq 0$ and $G \geq 0$ in D_1 . Hence by the fact that

---

(1) See [4], Chapter II.

$G = G_o - V$,

$$G(x,t;\xi,\tau;\varepsilon) \leq G_o(x,t;\xi,\tau;\varepsilon) \quad \text{in} \quad D_1+\partial D_1 \quad .$$

Now we shall show that $E(x,t;\xi,\tau;\varepsilon)$ is bounded in D_1 . By the restrictions (a) and (b), we have $w_{xx} \varepsilon C^1(D_1)$ and $E \varepsilon C^2(D_1)$. By the fact that $0 \leq G \leq G_o$ it suffices to find an upper bound for

$$\int_{D_1}\int G_o(x,t;\xi,\tau;\varepsilon)d\xi\, d\tau \quad .$$

We recall

$$G_o(x,t;\xi,\tau;\varepsilon) = \frac{1}{2\sqrt{\pi\varepsilon(t-\tau)}}\, e^{-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}} \quad .$$

Thus

$$\begin{aligned}
\int_{D_1}\int G_o d\xi\, d\tau &= \int_{D_1}\int \frac{1}{2\sqrt{\pi\varepsilon(t-\tau)}}\, e^{-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}}\, d\xi\, d\tau \\
&\leq \int_o^T \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\varepsilon(t-\tau)}}\, e^{-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}}\, d\xi\, d\tau \\
&= \int_o^T \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}\, e^{-\alpha^2}\, d\alpha\, d\tau
\end{aligned}$$

where we have let $\frac{x-\xi}{2\sqrt{\varepsilon(t-\tau)}} = \alpha$. Then by the fact that $\int_{-\infty}^{\infty} e^{-a^2} da = \sqrt{\pi}$,

we obtain

$$\int_{D_1}\int G_o d\xi\, d\tau \le \int_o^T d\tau = T \quad . \tag{1.12}$$

From (1.10) and (1.12), we get

$$\begin{aligned} |E(x,t;\varepsilon)| &= \left| \int_{D_1}\int G(x,t;\xi,\tau;\varepsilon) w_{\xi\xi} d\xi\, d\tau \right| \\ &\le \int_{D_1}\int G_o(x,t;\xi,\tau;\varepsilon)\, |w_{\xi\xi}|\, d\xi\, d\tau \\ &\le MT \end{aligned} \tag{1.13}$$

where $M = \max\limits_{\text{in } D_1} |w_{xx}|$. Hence it follows from (1.6) and (1.13) that

$$\lim_{\varepsilon \to 0} u(x,t;\varepsilon) = w(x,t)$$

uniformly for $(x,t) \in D_1$.

Note that if in (1.3) and (1.4) $\psi_1 \neq w$ and $\psi_2 \neq w$ on AC and BD respectively, i.e. arbitrary prescribed boundary conditions, then (1.10) will be replaced by

$$\begin{aligned} E(x,t;\varepsilon) = &\int_{AC} (\psi_1 - w) \frac{\partial G}{\partial \xi} d\tau - \int_{BD} (\psi_2 - w) \frac{\partial G}{\partial \xi} d\tau \\ &+ \int_{D_1}\int G\, w_{\xi\xi}\, d\xi\, d\tau \quad . \end{aligned} \tag{1.14}$$

Then in order to find the upper bound of $E(x,t;\varepsilon)$, we not only have to check the last integral of (1.14) as we have done above, but also $\int_{AC} \frac{\partial G}{\partial \xi} d\tau$ and $\int_{BD} \frac{\partial G}{\partial \xi} d\tau$. By the same token we check $\int_{AC} \frac{\partial G_o}{\partial \xi} d\tau$

and $\int_{BD} \frac{\partial G_o}{\partial \xi} d\tau$ instead.

$$\int_{AC} \frac{\partial G_o}{\partial \xi} d\tau = \int_o^T \frac{x-\xi}{4\sqrt{\pi}[\varepsilon(t-\tau)]^{3/2}} e^{-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}} d\tau . \qquad (1.15)$$

Let $\frac{x-\xi}{2\sqrt{\varepsilon(t-\tau)}} = \beta$. Then $d\beta = \frac{x-\xi}{4\sqrt{\varepsilon}(t-\tau)^{3/2}} d\tau$ and (1.15) becomes

$$\frac{1}{\varepsilon\sqrt{\pi}} \int_{\frac{x-\xi}{2\sqrt{\varepsilon t}}}^{\frac{x-\xi}{2\sqrt{\varepsilon}(t-T)}} e^{-\beta^2} d\beta . \qquad (1.16)$$

By the fact that

$$\lim_{\varepsilon \to o} \int_{\frac{x-\xi}{2\sqrt{\varepsilon t}}}^{\frac{x-\xi}{2\sqrt{\varepsilon(t-T)}}} e^{-\beta^2} d\beta = \sqrt{\pi} .$$

This with (1.15) and (1.16), we get

$$\int_{AC} \frac{\partial G_o}{\partial \xi} d\tau = 0(\frac{1}{\varepsilon}) \quad \text{as} \quad \varepsilon \to 0 .$$

Similarly ,

$$\int_{BD} \frac{\partial G_o}{\partial \xi} d\tau = 0(\frac{1}{\varepsilon}) \quad \text{as} \quad \varepsilon \to 0 .$$

In this case E is not bounded and we get a boundary layer phenomenon along AC and BD , which are the characteristics of the reduced equation. S.L. Kamenomostskaya [6] shows that on AC and BD there

is a boundary layer term z such that $|z| \leq K\, e^{-(g_i/\sqrt{\varepsilon})} + \sqrt{\varepsilon}\, Z$, $g_i > 0$ $(i = 1,2)$ near AC and BD , and $|z| \leq K\, e^{-(\delta/\sqrt{\varepsilon})} + \sqrt{\varepsilon}\, Z$, $\delta > 0$, elsewhere. Also $z = u - w$ on AC and BD .(1)

3. Removal of Restrictions

Now we shall relax the restrictions (a) and (b) in §2 to

(a)' $f \in C(D+\partial D)$, $f(x,t) \in C^1(D)$

(b)' $\phi(x) \in C^1(AB)$

as in the statement of the theorem.

Let $\{f_\nu(x,t)\}|_{\nu=1}^{\infty}$ be a sequence of function with $f_\nu \in C^2(D+\partial D)$ and $f_\nu \in C^3(D)$ such that $f_\nu \to f$ uniformly in $D+\partial D$ as $\nu \to \infty$. Let $\{\phi_\nu(x)\}|_{\nu=1}^{\infty}$ be a sequence of functions with $\phi_\nu \in C^3(AB)$ such that $\phi_\nu \to \phi$ uniformly in AB as $\nu \to \infty$. Let $w_\nu(x,t)$ solve

PDE	$w_t = -f_\nu$	in	$D_1+\partial D_1$
IC	$w = \phi_\nu$	on	AB .

Then

$$w_\nu = \phi_\nu - \int_o^t f_\nu(x,\tau)d\tau \quad .$$

Let $u_\nu(x,t;\varepsilon)$ solve

---

(1) This result is similar to V in §6 , Chapter III.

PDE	$\varepsilon\Delta u - u_t = f_\nu$	in $D_1+\partial D_1$
IC	$u = \phi_\nu$	on AB
BC	$u = w_\nu$	on AC and BD .

Then (i) $w_\nu \to w$ uniformly in $(D_1+\partial D_1)$ as $\nu \to \infty$

(ii) $u_\nu \to u$ uniformly in $(D_1+\partial D_1)$ as $\nu \to \infty$, and

(iii) $\lim_{\varepsilon \to o} u_\nu = w_\nu$ uniformly in D_1 .

Result (i):

$$|w_\nu - w| \leq |\phi_\nu - \phi| + \int_o^t (f_\nu - f)d\tau$$

$$\leq \Delta_\nu + \delta_\nu \cdot T \text{ , for } \nu > N$$

where $\Delta_\nu = \max\limits_{(x,t)\,\varepsilon\, AB} |\phi_\nu - \phi|$ and $\delta_\nu = \max\limits_{(\chi,t)\,\varepsilon\, D_1+\partial D_1} |f_\nu - f|$.

$$\therefore \quad \lim_{\nu \to \infty} w = w_\nu \quad \text{uniformly in } D_1+\partial D_1 \text{ .}$$

Result (ii):

Let $V = u_\nu - u$. Then V solves

PDE	$\varepsilon V_{xx} - V_t = f_\nu - f$	in $D_1+\partial D_1$
IC	$V = \phi_\nu - \phi$	on AB
BC	$V = w_\nu - w$	on AC and BD .

Then

$$V = \int_{AB} (\phi_\nu - \phi) G(x,t;\xi,\tau;\varepsilon) d\xi + \int_{AC} (w_\nu - w) \frac{\partial G}{\partial \xi} d\tau$$

$$+ \int_{BD} (w_\nu - w) \frac{\partial G}{\partial \xi} d\tau + \int_{D_1}\int (f_\nu - f) \cdot G \, d\xi \, d\tau \ .$$

For fixed $\varepsilon > 0$, we have that

$$\lim_{\nu \to \infty} V = 0 \ .$$

For this particular ν result (iii) follows from the proof in §2 .

Now to show that $u \to w$ uniformly as $\varepsilon \to 0$ for $(x,t) \in D_1$, we must show that for every $\delta > 0$, there exists a $\bar{\varepsilon} = \bar{\varepsilon}(\delta)$ such that $0 < \varepsilon < \bar{\varepsilon}$ implies that

$$|u - w| < \delta \qquad (x,t) \in D_1 \ .$$

So let us choose $\delta > 0$. Then

(i) choose $N_1(\delta)$ sufficiently large that $\nu > N_1$ implies that

$$|w_\nu - w| < \delta/3 \qquad (x,t) \in D_1 \ ,$$

(ii) choose $N_2(\delta)$ sufficiently large that $\nu > N_2$ implies that

$$|u_\nu - u| < \delta/3 \qquad (x,t) \in D_1 \ ,$$

(iii) choose $\bar{\varepsilon}(\delta)$ such that $0 < \varepsilon < \bar{\varepsilon}$ implies that

$$|u_\nu - w_\nu| < \delta/3 \qquad (x,t) \in D_1 \quad .$$

Hence for $\nu > N = \max\{N_1, N_2\}$ and $0 < \varepsilon < \bar{\varepsilon}$ we have

$$|u - w| \leq |u_\nu - u| + |u_\nu - w_\nu| + |w_\nu - w| < \delta \qquad (x,t) \in D_1 \quad .$$

CHAPTER VI

SOME QUESTIONS AND PROBLEMS

In conclusion we formulate some questions and problems which naturally arise from the preceeding considerations.

(1) We assume the functions in hypothesis H_1 in Chapter II are all C^6 and in hypothesis H_1 in Chapter III are all C^4. We could use Wasow's technique which is presented in Chapter IV to relax these restrictions.

(2) The problem of extending these methods to the same types of partial differential equations of $n^{\underline{th}}$ order in n dimensions for $n > 2$.

(3) It is desirable to pursue an investigation of the asymptotic solution u_ε of

$$L_\varepsilon[u] = \varepsilon(u_{xx} - u_{yy}) + au_x + bu_y + cu = d$$

in R, a region bounded by two characteristics of $L_\varepsilon[u] = f$ and two characteristics of the reduced equation

$$L_o[u] = aU_x + bU_y + cU = d \ ,$$

as shown in Figure 6.1 below, with the prescribed values $u_\varepsilon = U$ on I and $u_\varepsilon \neq U$ on J. It is clear that we should have a quite different maximum principle for $L_\varepsilon[u] = f$ from the maximum principles that we

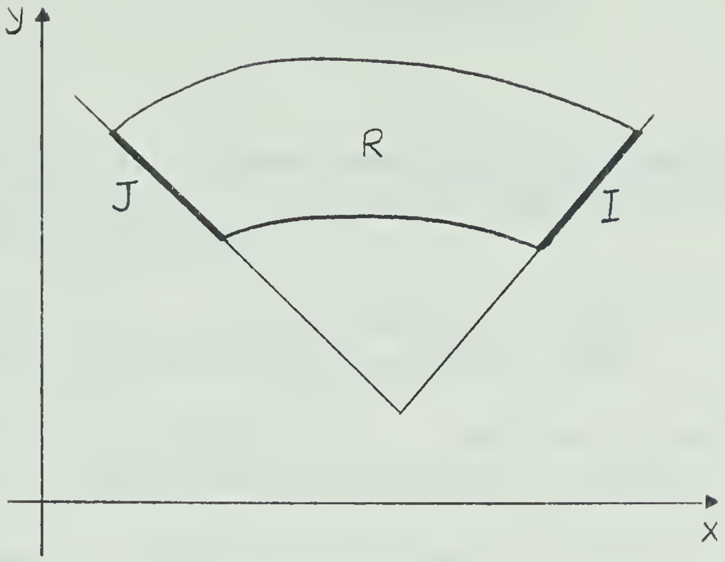

Figure 6.1

have for the elliptic and parabolic equations.

Then there naturally arises the question: "What happens when the characteristics of higher order equation and the characteristics of the lower order equation coincide?"

BIBLIOGRAPHY

1. Aronson, D.G., "Linear Parabolic Differential Equation Containing a Small Parameter." Journal of Mathematics and Mechanics, Vol. 5 (6)(1956), 1003-1014.

2. Coddington, E.A. and Levinson, N., Theory of Ordinary Differential Equations. McGraw-Hill, New York, 1955.

3. Courant, R. and Hilbert, D., Methods of Mathematical Physics, Vol. II. Interscience Publishers, New York, 1953.

4. Friedman, A., Partial Differential Equations of Parabolic Type. Prentice-Hall, Inc., New Jersey, 1964.

5. Friedrichs, K.O., "Asymptotic Phenomena in Mathematical Physics." Bulletin of American Mathematical Society, Vol. 61 (1955), 485-504.

6. Kamenomostskaya, S. L., "On Equations of Elliptic and Parabolic Type with a Small Parameter in the Highest Derivatives." Matematicheskii Sbornik, N.S. 31 (73)(1952), 703-708 (Russian).

7. Levinson, N., "The First Boundary Value Problem for $\varepsilon\Delta u + A(x,y)u_x + B(x,y)u_y + C(x,y)u = D(x,y)$ for small ε." Annals of Mathematics, Vol. 51 (1950), 428-445.

8. Lichtenstein, L., "Randwertaufgaben der Theorie der linearen partiellen Differentialgleichungen z weiter Ordnumg von elliptischen Typus I: Die erste Randwertaufgaben, Algemeine ebene Gebiete." J. Reine Angew. Math. 142 (1913), 1-40.

9. Tikhonov, A.N. and Samarskii, A.A., Equations of Mathematical Physics. Macmillan, New York, 1963.

10. Wasow, W., "Asymptotic Solution of Boundary Value Problems for the Differential Equation $\Delta u + \lambda u_x + \lambda f(x,y)$." Duke Mathematical Journal, Vol. 11 (1944), 405-415.